Testing a parametric transformation model versus a nonparametric alternative*

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Abstract

Despite an abundance of semiparametric estimators of the transformation model, no procedure has been proposed yet to test the hypothesis that the transformation function belongs to a finite-dimensional parametric family against a nonparametric alternative. In this paper we introduce a bootstrap test based on integrated squared distance between a nonparametric estimator and a parametric null. As a special case, our procedure can be used to test the parametric specification of the integrated baseline hazard in a semiparametric mixed proportional hazard (MPH) model. We investigate the finite sample performance of our test in a Monte Carlo study. Finally, we apply the proposed test to Kennan's strike durations data.

JEL: C12, C14, C41

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1 Introduction

Consider a transformation model of the form:

$$\Lambda_0(Y) = X'\beta_0 + U \tag{1}$$

where Y is a scalar dependent variable, X is a vector of q nondegenerate explanatory variables, β_0 is a vector of coefficients belonging to a compact set $\Theta_\beta \subset \mathbb{R}^q$, $\Lambda_0(\cdot)$ is an increasing function and U is an unobserved error term with cumulative distribution function F that is independent of X. For the model to be identified, the following normalizations are used: $\Lambda_0(y_0) = 0$ for some finite y_0 and $\beta_{0,1} = 1$ (where $\beta_{0,1}$ denotes the first element of β_0). Note that the model belongs to the class of single index models, therefore β_0 can be estimated $\sqrt{n}$-consistently using, for example, maximum rank correlation estimator (Han (1987)) or semiparametric least squares (Ichimura (1993)). We assume that such estimator is available throughout our analysis.

The main objective of this article is to provide a practically appealing test that would distinguish between various parametric specifications of the transformation function. Our procedure can be used to test a parametric form of the integrated baseline hazard function in duration models, to test the log-linear specification in wage regressions or the form of the marginal utility (profit) function in hedonic models (see Ekeland et al. (2004)). Among others, our procedure is relevant in experimental studies of demand elasticities (e.g. Jessoe & Rapson (2014), Hainmueller et al. (2015), Karlan & Zinman (2018)) where the dependent variable (sales, loan amounts, market shares etc.) is often transformed using $\log(\cdot)$ or $\log(\cdot + 1)$. Our test can be used to verify if such transformation is correctly specified and, thus, if these models provide correct elasticity estimates. Alternatively, if independence between X and U is not guaranteed by the experimental design, just as in Neumeyer et al. (2016), one can see our procedure as a general goodness-of-fit test for model (1).

Several nonparametric estimators have been proposed for the transformation function in model (1) with and without censoring: Horowitz (1996) (HJ henceforth), Gørgens & Horowitz (1999), Chen (2002) (CS henceforth), Ye & Duan (1997) and Klein & Sherman (2002). See also Linton et al. (2008) for estimation of a generalized model with nonparametric regression function. Despite such an abundance of semiparametric estimation techniques, the literature on testing parametric specification of the transformation model against an unrestricted one is small. A paper that is most closely related is Neumeyer et al. (2016). They propose to test the specification of the transformation by testing if a given parametric transformation is consistent with independence of X and U, so essentially their test is based on the estimated distribution of residuals, $\hat{U}$. The disadvantage of their test is the need to choose at least two bandwidths (see p. 939 in their paper) and additionally a tuning sequence for implementation of their smoothed bootstrap procedure and there is little practical guidance on how to choose these parameters in finite samples. As our procedure is based on rank estimators and bootstrapping it avoids the need of choosing tuning parameters. Specification testing in the quantile transformation model has been considered also by Mu & He (2007). There is also a related literature on testing single index models, see e.g. Härdle & Mammen (1993), Horowitz & Härdle (1994), Härdle et al. (1997), Neumeyer (2009).

Our test uses the nonparametric estimator of the transformation function developed by Chen (2002) and compares it to the parametric specification using the L^2 norm (or sup norm). We chose to build our test on this estimator for three reasons. Firstly, the CS estimator has a convenient linear asymptotic representation whereas no such representation is available for Klein & Sherman (2002) and Ye & Duan (1997), which makes the analysis of the test based on the latter estimators more complicated. Secondly, CS is much easier to compute than HJ since using the latter would involve multiple computationally intensive numerical integrations. Finally, as shown in Chen (2002) CS generally performs better than the other estimators in terms of root mean-square error, especially in the tails of the data distribution.

In our model F is treated nonparametrically. As an alternative to our approach, one can assume a parametric distribution for F, as in tests of Cox proportional hazard and mixed proportional hazard models in Horowitz & Neumann (1992), McCall (1994), Lin et al. (2006). Alternatively, if the data on Y is recorded on a finite grid, e.g. Y is unemployment duration and is recorded in weeks, then one can estimate Λ_0 at the points in the grid by maximum likelihood both with and without imposing a parametric restriction on Λ_0 and run a likelihood ratio test to verify if the parametric model is valid.[1] The disadvantage of these approaches is that misspecification of the parametric form of F may lead to invalid inference about the specification of Λ_0, whereas our approach will be

[1]See Meyer (1990) for estimation of an MPH model with nonparametric hazard, parametric distribution of U and discrete observations on Y.

robust to misspecifying F. Finally, our test can also be applied if F is restricted to a parametric class provided that the nonparametric estimator of Λ_0 satisfies the assumptions below.

The article is organized as follows. Section 2 discusses specification testing in the general transformation model given in (1). In this model the transformation function is identified only up to scale so inference boils down to checking if the shape of Λ_0 is consistent with the parametric assumption. Section 2.1 considers a special case of the mixed proportional hazard model. Thanks to additional structure, in this model both the shape and the scale of the transformation function are identified. We show that in order to test if the parametric specification of the integrated baseline hazard is correct it is enough to use the estimator up to scale. This has two advantages relative to simply comparing the estimated parametric and non-parametric integrated baseline hazards. Firstly, the scale of the integrated baseline hazard, whether in parametric or nonparametric models, can be estimated only at a rate slower than the standard $n^{-1/2}$ rate (see Hahn (1994), Ishwaran (1996)) so by using estimates up to scale we still obtain a test that has power against alternatives that are $O(n^{-1/2})$ apart from the null hypothesis. Second, available estimators of the scale (see Honoré (1990), Horowitz (1999)) are difficult to use in practice. For example, the estimator in the latter paper requires a choice of multiple tuning sequences converging to zero at appropriate rates, which is troublesome given lack of prescriptions for how to pick them in a finite sample.

Our test statistic converges to a functional of a Gaussian process and we suggest using bootstrap to obtain the critical values. We show that bootstrap consistently estimates the asymptotic distribution of our statistic. As a by-product of our analysis we prove that nonparametric bootstrap can be used to obtain standard errors for the CS estimator. This is an important result by itself since previous approaches based on numerical derivatives or kernel smoothing proved to be quite unstable and hard to implement in practice. In Section 3 we investigate the finite sample performance of our test using a Monte Carlo study. Section 4 provides an application to Kennan's strike duration data. Proofs are located in the Appendix and additional material is contained in the online supplement available at Cambridge Journals Online (journals.cambridge.org/ect.

2 General transformation model

We want to test:

$$H_0: \quad \Lambda_0(\cdot) \in \{\Lambda(\cdot,\gamma); \gamma \in \Theta_\gamma\} \quad \text{over} \quad [y_1, y_2]$$

where Θ_γ is an open subset of a d-dimensional Euclidean space. One needs to restrict oneself to a compact interval $[y_1, y_2]$ because $\Lambda_0(y)$ may not be bounded on the whole real line.[2] From now on we will refer to the model with parametric $\Lambda(\cdot,\gamma)$ as a 'parametric model' in contrast to a 'nonparametric model' in which Λ_0 is not restricted to lie in a parametric class, although both models leave the distribution of U unrestricted.

A natural way to construct a test is to take the L^2 (or sup) distance between one of the estimators $\Lambda_n(\cdot)$ and the parametric estimator, e.g. the estimator of Box-Cox regression model proposed by Foster et al. (2001). However, as mentioned in the Introduction, the transformation function is only identified up to scale and location normalizations. We have two cases. Firstly, the same normalization may be imposed on both nonparametric and parametric models, i.e. $\Lambda_0(y_0) = \Lambda(y_0, \gamma)$ for some $y_0 \in \mathbb{R}$, and $\beta_1 = 1$. Secondly, often a parametric model for the transformation imposes a scale normalization by itself so we cannot restrict $\beta_1 = 1$ (for example, if the parametric specification has a log-linear form: $\log Y = X'\beta + U$). Therefore, we have to normalize the nonparametric estimator so that the two transformation functions are comparable. This can be done by multiplying the nonparametric estimator by the estimator of the scale from the parametric model.[3]

Let $\hat{\beta}$ denote an estimator of the coefficient vector β in the parametric model and let $\hat{\beta}_1$ be its first element. Note that $\hat{\beta}_1\Lambda_n(y)$ is equal to the estimator of the transformation function when the normalization $\beta_{0,1} = \hat{\beta}_1$ is imposed instead of $\beta_{0,1} = 1$. Thus, our test statistic is given by:

$$T_n = n\int_{y_1}^{y_2} \left[(a_n\Lambda_n(y) - \Lambda(y,\hat{\gamma}))w(y)\right]^2 dy. \tag{2}$$

[2]One could expand the support of Λ with the sample size and as a result obtain a test over the whole support $\mathbb{R}$. We leave this extension for further research.

[3]Throughout the article we will use hats to denote the estimators obtained using the parametric model and subscript n to denote estimators corresponding to the nonparametric model.

where $\hat{\gamma}$ is an estimator of γ, $a_n = D + (1-D)\hat{\beta}_1$ and

$$D = \begin{cases} 1 & \text{if both transformations are normalized at the same point} \\ 0 & \text{otherwise} \end{cases}$$

The weight function $w(y)$ may be used to redirect the power of the test over y. For example, an application may dictate that some region of y's is of particular interest.

In principle, instead of using a Cramér-von-Mises type test, a Kolmogorov-Smirnov type test can be used. The latter test would have more power against sharp-peaked alternatives. As our proofs involve showing uniform convergence of the integrand in (2) it is straightforward to extend our results to the Kolmogorov-Smirnov test.[4]

Frequently, especially in the context of duration models, the observations on Y_i are right-censored. Let C_i denote a random censoring threshold with cumulative distribution function G_0 and survival function $\bar{G}_0$, let $\tilde{Y}_i$ denote a latent (not censored) value of the dependent variable generated from (1) and let Y_i be a censored observation on $\tilde{Y}_i$, i.e. $Y_i = \min\{\tilde{Y}_i, C_i\}$. Additionally, define a censoring indicator $\delta_i = \mathbb{1}\{\tilde{Y}_i \leq C_i\}$.

From now on, we will focus on the case in which Y's are censored. The case without censoring can be seen as a special case with $C_i = \infty$ for all i (i.e. $\bar{G}_0(y) = 1$ for all y in $[y_1, y_2]$) so all the arguments below will apply to this special case.

Define the Euclidean class of functions as in Pakes & Pollard (1989) and let $L^2(\mathcal{Y})$ denote a space of square integrable functions on $\mathcal{Y}$. We make the following assumptions:

Assumption 1. (DGP) *$\{X_i, Y_i, \delta_i : i = 1, \ldots, n\}$ is a random sample, U is independent of X, C is independent of (X, U) and $\bar{G}_0$ is bounded away from zero on $[y_1, y_2]$.*

Assumption 2. (Asymptotic linearity)

(a) There is a function $J : [y_1, y_2] \times \mathbb{R}^q \times \{0,1\} \times [y_1, y_2] \times \Theta_\beta \to \mathbb{R}$ such that $E[J(Y, X, \delta; y, \beta_0)] = 0$, $E[J(Y, X, \delta; y, \beta_0) J(Y, X, \delta; y', \beta_0)]$ is finite for every $y, y' \in [y_1, y_2]$, $J(Y, X, \delta; \cdot, \beta_0) \in L^2([y_1, y_2])$

[4]See Online Appendix B.1 for Monte Carlo simulations with the KS test.

and, as $n \to \infty$*:*

$$\sqrt{n}(\Lambda_n(y) - \Lambda_0(y)) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} J(Y_i, X_i, \delta_i; y, \beta_0) + o_p(1)$$

uniformly over $y \in [y_1, y_2]$*. Moreover, the class of functions* $\mathcal{J} = \{J(\cdot, \cdot, \cdot; y, \beta_0), y \in [y_1, y_2]\}$ *is Euclidean.*

(b) Let γ *be a probability limit of* $\hat{\gamma}$*. There exists a vector-valued function* $\Omega_\gamma(Y_i, X_i, \delta_i; \gamma, \beta)$ *with mean zero and finite covariance matrix such that, as* $n \to \infty$*:*

$$\sqrt{n}(\hat{\gamma} - \gamma) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Omega_\gamma(Y_i, X_i, \delta_i; \gamma, \beta) + o_p(1).$$

(c) $\Lambda(y, \gamma)$ *is twice differentiable in* γ *and the derivatives are bounded uniformly over* $y \in [y_1, y_2]$*.*

(d) Let β_1 *be a probability limit of* $\hat{\beta}_1$*. There exists a function* $\Omega_1(Y_i, X_i, \delta_i; \gamma, \beta)$ *with mean zero and finite variance such that, as* $n \to \infty$*:*

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Omega_1(Y_i, X_i, \delta_i; \gamma, \beta) + o_p(1).$$

Assumption 3. (Weight function) *The weight function* $w(y)$ *satisfies:* $\int_{y_1}^{y_2} w(y)^2 dy = 1$.

Assumption 2(a) is satisfied by the CS and by the HJ estimator.[5] This assumption implies that $\sqrt{n}(\Lambda_n(y) - \Lambda_0(y))$ converges to a mean zero Gaussian process. Assumptions 2(b),(c) are not relevant if $\Lambda(y, \gamma)$ does not depend on γ as in our leading example of testing a log-linear model versus a nonparametric alternative, i.e. $\Lambda(y, \gamma) = \log(y)$. Assumption 2(b) is satisfied by the estimator proposed in Foster et al. (2001)[6] and the estimators for Box-Cox and Bickel-Doksum parameters suggested in Han (1987) and analyzed in Asparouhova et al. (2002) (see Online Appendix D). Assumption 2(c) is satisfied by the Box-Cox transformation (with $y_1 > 0$) and most hedonic pricing models if the utility (profit) function is sufficiently smooth (e.g. Cobb-Douglas). The asymptotic

[5] Klein & Sherman (2002) only show point-wise convergence of their estimator to a normal variable. They do not provide a uniform linear representation as in Assumption 2(a). Also the estimator developed by Ye & Duan (1997) does not have a linear representation.

[6] Foster et al. (2001) include an intercept in their model and set $E(U) = 0$. This is in line with our model in (1) as we do not restrict the mean of U.

linear representation in Assumption 2(d) is clearly available for the OLS estimator in the loglinear model but also for the estimator developed by Foster et al. (2001) for the Box-Cox model.

Example 1. ***(log-linear model)*** *We test if a wage regression has a log-linear form. For simplicity assume that there is only one regressor and no censoring. We estimate the model by ordinary least squares. In this case we have* $\Lambda(y,\gamma) = \log(y)$, $\frac{\partial \Lambda(y,\gamma)}{\partial \gamma} = 0$ *and* $\Omega_1(Y_i, X_i; \gamma, \beta) = (X_i - \bar{X})(\log Y_i - \beta X_i)/Var(X_i)$.

Define:

$$B_n(y) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [(D + (1-D)\beta_1) J(Y_i, X_i; y, \beta_0) - \frac{\partial \Lambda(y,\gamma)}{\partial \gamma}' \Omega_\gamma(Y_i, X_i; \gamma, \beta) + (1-D)\Lambda(y,\gamma)\Omega_1(Y_i, X_i; \gamma, \beta)] w(y). \qquad (3)$$

The following theorem establishes the asymptotic approximation to the distribution of the test statistics.

Theorem 1. *Under* H_0 *and Assumptions 1-3:*

$$T_n \to^d \int_{y_1}^{y_2} \mathcal{B}^2(y) dy \qquad (4)$$

where $\mathcal{B}$ *is a mean zero Gaussian process with covariance function* $R(y, y') = E[B_n(y)B_n(y')]$. *Alternatively, we can write:*

$$T_n \to^d \sum_{j=1}^{\infty} \omega_j \chi^2_{j1}, \qquad (5)$$

where χ^2_{j1}*'s are independent chi-square random variables with one degree of freedom and* ω_j*'s are eigenvalues of the linear integral operator:*

$$(\mathcal{R}g)(y) = \int_{y_1}^{y_2} R(y,z) g(z) dz; \qquad g(\cdot) \in L^2([y_1, y_2]). \qquad (6)$$

The alternative formulation in (5) follows from principal component decomposition of $\mathcal{B}(y)$, just as in Durbin & Knott (1972), and will be useful for analyzing local power of our test.

We can obtain the critical value for our test by simulating the process $\mathcal{B}$ and calculating the integral in (4).[7] However, this would require estimating the covariance function R which both for CS and HJ estimators involves kernel smoothing. Since there are no procedures to choose a bandwidth for these estimators in the finite sample and, as evidenced by our simulation studies (available upon request), the results of the test are very sensitive to this choice, we do not pursue this approach. Instead, we suggest using bootstrap critical value.

2.1 MPH duration model

Before we turn to the bootstrap procedure, we briefly discuss how our test can be used to test a parametric specification of the (integrated) baseline hazard in duration models. A duration model can be seen as a special case of the transformation model. We consider the single-spell mixed proportional hazard (MPH) model:

$$\alpha \log \tilde{\Lambda}(Y) = X'\beta + V - \xi \tag{7}$$

where $\tilde{\Lambda}(Y)^{\alpha}$ is the integrated baseline hazard, ξ has the standard Gumbel distribution and (ξ, V, X) are mutually independent. For simplicity, there is no censoring. We intentionally factored out the scale of the log of integrated hazard, α, to facilitate discussion below. The difference between this model and the general transformation model discussed before is that here β and α are separately identified and we do not need the normalization $\beta_1 = 1$.

Now observe that, if $\tilde{\Lambda}$ is known, equation (7) pins down the scale α because the scale of ξ is fixed (and ξ is independent of X and V). In other words, if there are two MPH models:

$$\alpha^{(1)} \log \tilde{\Lambda}^{(1)}(Y) = X\beta^{(1)} + V^{(1)} - \xi$$
$$\alpha^{(2)} \log \tilde{\Lambda}^{(2)}(Y) = X\beta^{(2)} + V^{(2)} - \xi$$

[7] As an alternative, one can use the characterization in (5) and employ the simulation procedure in Horowitz (2006) and Blundell & Horowitz (2007). This would involve truncating the sum in (5) and estimating the remaining eigenvalues ω_j. This is straightforward in the setting analyzed by Horowitz (2006) and Blundell & Horowitz (2007) because the Fourier representation of the covariance kernel can be calculated analytically without numerical integration. This is not the case here since the covariance function includes an at least three dimensional non-separable function $J(\cdot, \cdot; \cdot, \beta_0)$, which entails the need to perform a triple numerical integration in order to obtain the Fourier coefficients. This makes this method unattractive in our setting.

with $\tilde{\Lambda}^{(1)} = \tilde{\Lambda}^{(2)}$, then they can generate the same population distribution of Y given X only if $\alpha^{(1)} = \alpha^{(2)}$ (excluding a knife-edge case when $V^{(1)}/\alpha^{(1)}$ and $V^{(2)}/\alpha^{(2)}$ have the same distribution as $-\xi$). As a result, if we want to test if the integrated baseline hazard $\alpha \log \tilde{\Lambda}(\cdot)$ belongs to some parametric class, it is enough to test that the estimate up to scale, $\log \tilde{\Lambda}(\cdot)$, belongs to a conjectured parametric family. Therefore, the following procedure can be used:

1. Estimate the transformation model in (1) imposing the necessary normalizations.

2. Estimate the null parametric transformation $\Lambda(y,\gamma) = \log \tilde{\Lambda}(y,\gamma)$ with (this would correspond to $D = 0$ above) or without ($D = 1$) imposing the normalization $\beta_1 = 1$, for example by using GMM (see Horowitz (2009), Ch. 6.1) or Foster et al. (2001).

3. Run our bootstrap test (see next section for details). If the test statistic is greater than the critical value, conclude that the integrated baseline hazard is misspecified.

This is convenient since estimators of α do not converge at the $n^{-1/2}$ rate either in the parametric or nonparametric model. For the Weibull MPH model Honoré (1990) shows that under the assumption $E[e^{-V}] < \infty$ his estimator converges at a rate that can be made arbitrarily close to $n^{-1/3}$. As shown by Ishwaran (1996), the highest rate at which an estimator of α converges to the true value under the assumption $E[e^{-V}] < \infty$ is $n^{-1/3}$, and $n^{-2/5}$ under the assumption $E[e^{-3V}] < \infty$. On the other hand, estimators of $\log \tilde{\Lambda}(\cdot)$ converge at the usual $n^{-1/2}$ rate. Thus, by avoiding the need to estimate the scale α in our test we sustain this fast rate of convergence.

Example 2. (Weibull MPH model) *We test if the integrated baseline hazard has a Weibull shape, i.e. if* $\log \tilde{\Lambda}(y) = \log(y)$*. Now the MPH model becomes:*

$$\log(Y) = X'\frac{\beta}{\alpha} + \frac{V - \xi}{\alpha}$$

and $\tilde{\beta}_1 = \beta_1/\alpha$ *can be estimated* $\sqrt{n}$*-consistently by OLS. We can use* $\hat{\tilde{\beta}}_1$ *as the scaling factor. Since the transformation function does not depend on unknown parameters, the second term in the expression for* B_n *(equation* (3)*) vanishes.*

2.2 Bootstrap critical value

The theory developed so far applies both to the HJ and CS estimators. Nevertheless, CS is preferred from the computational point of view. Using HJ to compute the test statistic involves double numerical integration to obtain Λ_n on top of the integration involved in computing the L^2 distance. Doing that repetitively to obtain the bootstrap critical value would entail a very large computational cost. It is much easier to bootstrap the CS estimator. Thus, from now on we will assume that Λ_n is the CS estimator. In the case of censored observations this estimator is defined as:

$$\Lambda_n(y) = \arg\max_{\Lambda} \frac{1}{n(n-1)} \sum_{i \neq j} \left(\frac{d_{iy}}{\bar{G}_n(y)} - \frac{d_{jy_0}}{\bar{G}_n(y_0)} \right) \mathbb{1}\{Z_i - Z_j \geq \Lambda\} \tag{8}$$

where $\bar{G}_n(y)$ is the Kaplan-Meier estimator of the survival function of the censoring threshold C and $d_{iy} = 1\{Y_i \geq y\}$, $d_{jy_0} = 1\{Y_j \geq y_0\}$ and $Z_i = X_i' b_n$.

Let $w_1 = (x^1, y^1)$ and $w_2 = (x^2, y^2)$. Define:

$$r(w_1, w_2, y, G, \Lambda, b) = \left(\frac{\mathbb{1}\{y^1 \geq y\}}{\bar{G}(y)} - \frac{\mathbb{1}\{y^2 \geq y_0\}}{\bar{G}(y_0)} \right) (\mathbb{1}\{x^1 b - x^2 b \geq \Lambda\} - \mathbb{1}\{x^1 b - x^2 b \geq \Lambda_0\})$$

and:

$$\tau(w, y, \Lambda) = E[r(w, W, y, G_0, \Lambda, \beta_0) + r(W, w, y, G_0, \Lambda, \beta_0)]$$

for $W = (X, Y)$. Let:

$$V(y) = E\left[-\frac{\partial^2 \tau(W, y, \Lambda)}{\partial \Lambda^2} \bigg|_{\Lambda = \Lambda_0} \right].$$

Finally, let X_1 be the first component of X and X_{-1} denote the remaining $(q-1)$ components.

Our bootstrap procedure will be valid under assumptions similar to those introduced in Chen (2002) and Jochmans (2012):

Assumption 4. (Chen (2002))

(a) The normalization $\beta_1 = 1$ is imposed on the nonparametric estimator Λ_n.

(b) The distribution of X_1 conditional on $X_{-1} = x_{-1}$ is absolutely continuous with respect to the

Lebesgue measure.

(c) *The support of X is not contained in any proper linear subspace of $\mathbb{R}^q$.*

(d) $\Lambda_0(\cdot)$ *is strictly increasing,* $\Lambda_0(y_0) = 0$, $[\Lambda_0(y_1 - \varepsilon), \Lambda_0(y_2 + \varepsilon)] \subset \Theta_\Lambda$ *for a small positive number* ε, *where* Θ_Λ *is a compact interval.*

(e) *The conditional density of* X_1 *given* $X_{-1} = x_{-1}$ *and the density of* U *are bounded and twice continuously differentiable, the derivatives are uniformly bounded and* X_{-1} *has finite third-order moments.*

(f) $V(y)$ *is positive for each* $y \in [y_1, y_2]$ *and uniformly bounded away from zero.*

(g) *The first step estimator of* β_0 *from the nonparametric model,* b_n, *has the following asymptotic representation:*[8]

$$\sqrt{n}(b_n - \beta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Omega^{NP}(Y_i, X_i, \delta_i; \beta_0) + o_p(1).$$

where Ω^{NP} *is a mean zero vector valued function with finite variance-covariance matrix.*

We will employ the following bootstrap procedure to obtain a critical value for our test:

1. Draw a random sample $\{(Y_i^*, X_i^*, \delta_i^*) : i = 1, \ldots, n\}$ with replacement from $\{(Y_i, X_i, \delta_i) : i = 1, \ldots, n\}$ or use a parametric bootstrap:

    - Estimate $(\hat{\beta}, \hat{\gamma})$ using $\{(Y_i, X_i) : i = 1, \ldots, n\}$.
    - Generate $\hat{U}_i = \Lambda(Y_i, \hat{\gamma}) - X_i'\hat{\beta}$.
    - Draw a random sample $\{U_i^* : i = 1, \ldots, n\}$ with replacement from $\{\hat{U}_i : i = 1, \ldots, n\}$ and calculate $Y_i^* = \Lambda^{-1}(X_i'\hat{\beta} + U_i^*, \hat{\gamma})$.

2. Using the bootstrap sample calculate $(\hat{\beta}_1, \hat{\gamma})$ from the parametric model and (Λ_n, b_n) from the nonparametric model. Let the resulting estimates be denoted by $(\hat{\beta}_1^*, \hat{\gamma}^*)$ and (Λ_n^*, b_n^*).

[8]Recall that the estimator obtained from the model with parametric Λ is denoted by $\hat{\beta}$. We can have $\hat{\beta}_1 \neq 1$ whereas $b_{n1} = 1$ by assumption.

3. Calculate the bootstrap statistic:

$$T_n^* = n \int_{y_1}^{y_2} \left[(a_n^* \Lambda_n^*(y) - a_n \Lambda_n(y) - (\Lambda(y, \hat{\gamma}^*) - \Lambda(y, \hat{\gamma})))w(y) \right]^2 dy.$$

if nonparametric bootstrap has been used, or:

$$T_n^* = n \int_{y_1}^{y_2} \left[(a_n^* \Lambda_n^*(y) - \Lambda(y, \hat{\gamma}^*))w(y) \right]^2 dy.$$

for parametric bootstrap, where $a_n^* = D + (1 - D)\hat{\beta}_1^*$.

4. Obtain the empirical distribution of T_n^* by repeating steps 1-3 many times. Calculate the $1-\kappa$ quantile of this empirical distribution. Denote it by c_κ^*.

If data are not censored, then we recommend to use parametric bootstrap as it usually leads to better performance in a finite sample. On the other hand, applying parametric bootstrap is complicated with censored data so we prefer nonparametric bootstrap in this case. Finally, note that the statistic corresponding to the parametric bootstrap does not require recentering as the parametric bootstrap imposes the null hypothesis contrary to nonparametric resampling.

On top of the assumptions above we will need an asymptotic linear approximation in the bootstrap sample:[9]

Assumption 5. (Bootstrap asymptotic linearity) *We have:*

$$E \left| \hat{\gamma}^* - \gamma - \frac{1}{n} \sum_{i=1}^{n} \Omega_\gamma(Y_i^*, X_i^*, \delta_i^*; \gamma, \beta) \right| = o(n^{-1/2}) \tag{9}$$

$$E \left| \hat{\beta}_1^* - \beta_1 - \frac{1}{n} \sum_{i=1}^{n} \Omega_1(Y_i^*, X_i^*, \delta_i^*; \gamma, \beta) \right| = o(n^{-1/2}) \tag{10}$$

$$E \left| b_n^* - \beta_0 - \frac{1}{n} \sum_{i=1}^{n} \Omega^{NP}(Y_i^*, X_i^*, \delta_i^*; \beta_0) \right| = o(n^{-1/2}) \tag{11}$$

where $\Omega_\gamma, \Omega_1, \Omega^{NP}$ have zero mean and finite variance-covariance matrix.

In the leading case when the parametric model in the null hypothesis does not depend on any free parameters (e.g. testing the Weibull model in duration analysis), condition (9) is redundant. In

[9]The meaning of the expectation operator here is explained in the Appendix where we formally define the probability space for handling joint sample and bootstrap randomness.

the next section we show that this condition is satisfied for the estimator in Chen (2012). Condition (10) will be satisfied for the OLS estimator. An asymptotic bootstrap linear representation for the rank estimators b_n introduced in Han (1987), Cavanagh & Sherman (1998) and Abrevaya (2003) follows from Subbotin (2007).

The following theorem states that the bootstrap critical value gives the correct approximation to the asymptotic critical value:

Theorem 2. *Under H_0 and Assumptions 1, 2*(b)-(d)*,3-5:*

$$\lim_{n\to\infty} P(T_n \leq c_\kappa^*) = 1 - \kappa.$$

The proof of this theorem is similar to the results in Subbotin (2007) who proves bootstrap validity for rank estimators. A complication in the proof compared to his work comes from the fact that the rank objective function in (8) contains estimators b_n and $\bar{G}_n$, which will contribute to the asymptotic distribution of Λ_n and Λ_n^*. Our proof also works under slightly weaker conditions than his. The argument leading to Theorem 2 implies also a following useful corollary:

Corollary 1. *Let P_n^* denote conditional probability given the sample $\{X_i, Y_i, \delta_i : i = 1, \ldots, n\}$. If Assumptions 1, 2*(a)*, 4 and condition* (11) *in Assumption 5 hold, then nonparametric bootstrap approximates consistently the asymptotic distribution of the CS estimator, i.e.*

$$\sup_{t\in\Theta_\Lambda} \sup_{y\in[y_1,y_2]} |P_n^*(\Lambda_n^*(y) \leq t) - P(H_\Lambda(y) \leq t)| = o_p(1)$$

where H_Λ is the Gaussian process defined in Theorem 2 in Chen (2002).

This result is important because it provides an operational method for obtaining standard errors for the CS estimator. Previous approaches based on numerical derivatives or kernel smoothing relied on arbitrary choices of the approximation step or bandwidth with the results being very sensitive to inappropriate choices of these tuning parameters. On the downside, bootstrapping the CS estimator is computationally costly, but not prohibitively so as shown by our MC simulations and application.

2.3 Bootstrap asymptotic linear approximation for censored semiparametric Box-Cox model

We will verify that Assumption 5 holds for the estimators of γ and β in the censored Box-Cox transformation model proposed by Chen (2012) (we analyze a model without censoring in Online Appendix E). We observe $Y_i = \max\{\tilde{Y}_i, c\}$ where c is a known censoring constant.[10] The Box-Cox transformation is given by:[11]

$$\Lambda(y,g) = \begin{cases} \frac{y^g - 1}{g} & \text{if } g \neq 0 \\ \log y & \text{otherwise} \end{cases}$$

Chen (2012) suggests to estimate (γ, β) using a two-step estimator:

1. First, for any candidate g estimate a curve $\hat{\beta}(g)$ by minimizing:

$$S_n(g,b) = \frac{1}{n(n-1)} \sum_{i \neq j} s(\Lambda(Y_i,g) - \Lambda(c,g), \Lambda(Y_j,g) - \Lambda(c,g), (X_i - X_j)'b)$$

with respect to b, where:

$$s(y_1,y_2,\Delta) = \begin{cases} y_1^2 - 2(y_2+\Delta)y_1 & \text{if } \Delta \leq -y_2 \\ (y_1 - y_2 - \Delta)^2 & \text{if } -y_2 < \Delta < y_1 \\ y_2^2 + 2(\Delta - y_1)y_2 & \text{if } y_1 \leq \Delta \end{cases}$$

2. Then estimate γ by maximizing:

$$R_n(g,\hat{\beta}(g)) = \frac{1}{n(n-1)} \sum_{i \neq j} \int_c^{\infty} \int_c^{\infty} (\mathbb{1}\{Y_i < y_1\} - \mathbb{1}\{Y_j < y_2\}) \times \mathbb{1}\{\Lambda(y_1,g) - \Lambda(y_2,g) \geq (X_i - X_j)'\hat{\beta}(g)\} d\Psi_1(y_1) d\Psi_2(y_2)$$

where $\Psi_1(y), \Psi_2(y)$ are differentiable, strictly increasing, deterministic and bounded weight

[10] We follow directly the convention in Chen (2012) and consider censoring from below here. The estimator can be easily modified to deal with censoring from above and our results apply in this case with only straightforward modifications in our assumptions.

[11] As mentioned in Chen (2012) his estimator can also be applied to generalizations of the Box-Cox transformation in Bickel & Doksum (1981).

functions. Let $\hat{\gamma}$ denote this estimator. Estimate β by $\hat{\beta}(\hat{\gamma})$.

Define $\beta(g)$ to be a minimizer of $E[S_n(g,\beta)]$ for given g. Let $\theta = (g, b) \in \Theta$, $\theta_0 = (\gamma, \beta)$ and θ^* be the corresponding estimators calculated on the bootstrap sample. Now with $w_l = (x^l, y^l), l = 1, 2$ and $y = (y_1, y_2)$ define:

$$\begin{aligned} h^{BC}_{\theta,y}(w_1, w_2) = {} & (\mathbb{1}\{y^1 < y_1\} - \mathbb{1}\{y^2 < y_2\})\mathbb{1}\{\Lambda(y_1, g) - \Lambda(y_2, g) \geq (x^1 - x^2)'b\} \\ & - (\mathbb{1}\{y^1 < y_1\} - \mathbb{1}\{y^2 < y_2\})\mathbb{1}\{\Lambda(y_1, \gamma) - \Lambda(y_2, \gamma) \geq (x^1 - x^2)'\beta\} \end{aligned}$$

and $\tau_{BC}(w, y, \theta) = E\left[h^{BC}_{\theta,y}(w, W) + h^{BC}_{\theta,y}(W, w)\right]$, where the expectation is taken with respect to $W = (X, Y)$. Then define:

$$V_{BC} = E\left[\int_c^\infty \int_c^\infty \partial^2 \tau_{BC}(W, y, \theta_0) d\Psi_1(y_1) d\Psi_2(y_2)\right]$$

with $\partial^2 \tau_{BC}(w, y, \theta)$ denoting the matrix of second derivatives of $\tau_{BC}(w, y, \theta)$ with respect to θ.

Theorem 3. *Let Assumptions 4*(b),(c),(e) *hold. Furthermore, assume:*

(a) $\Psi_1(y)$ *and* $\Psi_2(y)$ *are supported on a compact interval* $\mathcal{Y} \subset [c, \infty)$, $\Theta = \Theta_\gamma \times \Theta_\beta$ *is compact and* (γ, β) *is an interior point of* Θ,

(b) $P(Y_i > c|X_i) > 0$ *for almost every* X_i,

(c) There exists a small neighborhood of γ, $\mathcal{N}_\gamma$, *such that:*

$$E\left[\sup_{g \in \mathcal{N}_\gamma} \left|\frac{\partial^2 \Lambda(Y, g)}{\partial g^2}\right|\right]^2 < \infty \qquad \textit{and} \qquad E\left[\sup_{g \in \mathcal{N}_\gamma} \left|\frac{\partial^2 \phi_1(X, Y, g)}{\partial g^2}\right|\right]^2 < \infty$$

where ϕ_1 *is defined in Chen (2012),*

(d) V_{BC} *is a negative definite matrix,*

then Assumption 5 is satisfied for the estimators of (γ, β_1) *introduced in Chen (2012).*

The proof follows similar lines to the proof of Theorem 2 and is given in the Appendix. Conditions of the theorem imply that Assumptions 1-4 in Chen (2012) hold. For example, our Assumption 4(e) implies his Assumption 4(b).

2.4 Consistency and behaviour under local alternatives

We conclude this section with an analysis of power and local behaviour of our bootstrap test. Assume that the null hypothesis is false, i.e. there is no $\gamma \in \Theta_\gamma$ such that $\Lambda_0(\cdot) = \Lambda(\cdot, \gamma)$ a.e. Define:

$$q(y) = \Lambda_0(y) - \Lambda(y, \gamma)$$

where γ is a probability limit of $\hat{\gamma}$. The following theorem establishes consistency of the test under a fixed alternative:[12]

Theorem 4. *Let Assumptions 1, 2*(b)-(d)*,3-5 hold. Additionally, let H_0 be false and $\int_{y_1}^{y_2}[q(y)w(y)]^2dy > 0$. Then, for $\kappa \in (0,1)$ we have:*

$$\lim_{n\to\infty} P(T_n > c^*_\kappa) = 1.$$

Here and in the next theorem the values γ and β_1 (from the parametric model) described in Assumptions 2(b),(d) are interpreted as pseudo true values because the parametric model is misspecified.

Now consider local alternatives of the form:

$$\Lambda(y) = \Lambda(y, \gamma) + \frac{1}{\sqrt{n}}\Lambda^{loc}(y), \tag{12}$$

where $\Lambda(y, \gamma) = \Lambda_0(y)$ and $\Lambda^{loc}(\cdot) \in L^2([y_1, y_2])$. Let the sequence of functions $\{\psi_j\}_{j=0}^{\infty}$ form an orthonormal basis of $L^2([y_1, y_2])$. The following theorem provides local asymptotics:

Theorem 5. *Let Assumptions 1, 2*(b)-(d)*,3-5 hold. Under the sequence of local alternatives described in* (12)*:*

$$T_n \to^d \sum_{j=1}^{\infty} \omega_j \chi^2_{j1}\left(\frac{\vartheta_j^2}{\omega_j}\right),$$

where $\vartheta_j = \int_{y_1}^{y_2} \Lambda^{loc}(y)w(y)\psi_j(y)dy$ and $\chi^2_{j1}\left(\frac{\vartheta_j^2}{\omega_j}\right)$ denotes a noncentral chi-square random variable with 1 degree of freedom and noncentrality parameter $\frac{\vartheta_j^2}{\omega_j}$.

[12]See Online Appendix for proofs of the results in this Section.

Theorem 5 implies that the test has local power against local alternatives that are $n^{-1/2}$ away from the null hypothesis. In principle, different choices of the nonparametric estimator Λ_n will yield different eigenvalues ω_j and thus different local power. However, it is difficult to compare them theoretically since the kernels of the operator $\mathcal{R}$ in (6) for different estimators (HJ, CS and Ye & Duan (1997)) are complicated functions of y. The eigenvalues are usually computed as solutions to differential equations that involve derivatives of the kernels. Hence, the general expressions are hard to get.

3 Monte Carlo simulations

We investigate finite sample performance of the aforementioned testing procedures using several simple designs. We consider both the case when the model in the null hypothesis does not (linear transformation) and does (Box-Cox transformation) depend on the unknown parameter.

3.1 Linear transformation

The data is generated from the following three models:

$$Y = X + U \qquad \text{(Null)}$$

$$\log(Y + 2.12) - \log(2.12) = X + U \qquad \text{(Alternative 1)}$$

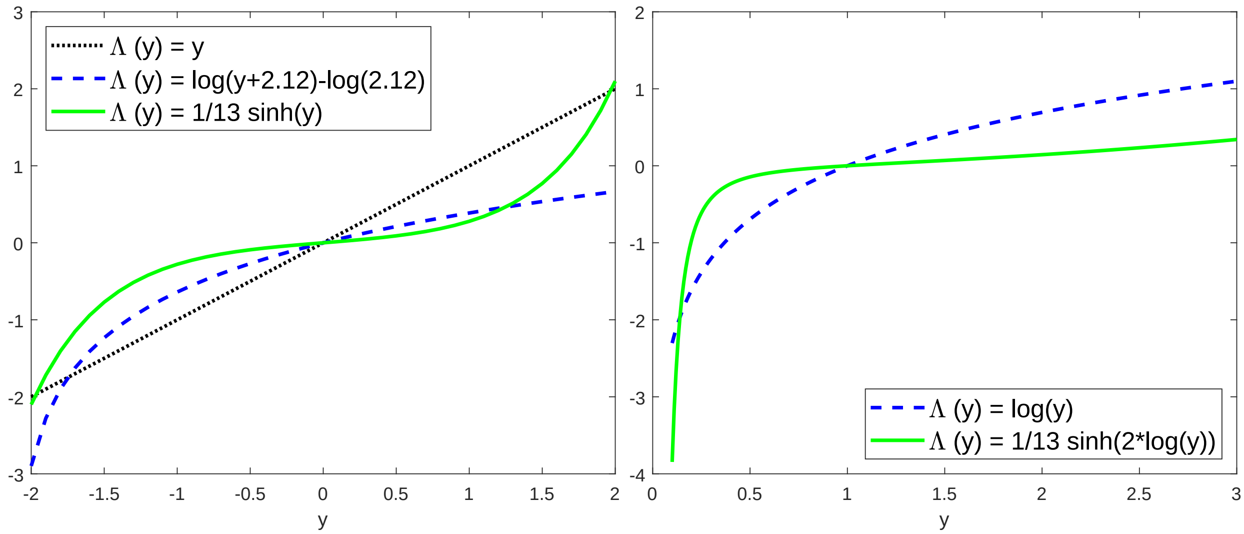
$$\frac{1}{13}\sinh(2Y) = X + U \qquad \text{(Alternative 2)}$$

where X is drawn from the standard normal distribution and U is drawn either from the standard normal, the standard Gumbel or from the logistic distribution. We shifted the logarithmic function by 2.12 in order to minimize L^2 distance of the logarithmic transformation in Alternative 1 to the linear function in the null. We set $[y_1, y_2] = [-2, 2]$.[13] The transformation functions under the null and under the alternatives are normalized at the same point $y_0 = 0$ (though, we do not use this information for running our test i.e. $D = 0$). This design is similar to the one used in Horowitz (1996). Figure 1 (left panel) shows the shape of the transformation functions.

The model with logistic U can be interpreted as an MPH model with V having the standard

[13] Online Appendix B.2 contains results with $[y_1, y_2] = [-3, 3]$ and $[y_1, y_2] = [-4, 4]$ in a model with no censoring. The results are very similar to the ones in Table 1.

Figure 1: Monte Carlo designs: linear (left panel) and Box-Cox (right panel) models



Gumbel distribution. Under this interpretation the null model assumes an increasing baseline hazard $\lambda(y) = e^y$, Alternative 1 implies that this hazard is constant and in Alternative 2 the baseline hazard equals $\frac{2}{13}\cosh(2y)e^{\frac{1}{13}\sinh(2y)}$ and is non-monotonic.

We consider both the case when Y is fully observed as well as the case when Y is randomly censored. In the former case we use parametric bootstrap. In the latter case the censoring threshold C is drawn from $N(\mu, 1)$ and μ is chosen such that the probability of being censored is roughly equal to 20%. The coefficient vector β is either estimated by OLS or RCLAD estimator of Honoré et al. (2002).

We run 2000 Monte Carlo replications. We calculate the integral in the test statistic using Halton sequences of size 100. Optimization needed to compute the nonparametric estimator Λ_n was performed using the Nelder-Mead simplex algorithm. The starting values for the optimization were taken from the null model whether the data was generated by this model or the alternative. The number of bootstrap replications used to calculate the critical value is 500. One Monte Carlo replication in the case with no censoring takes 2.1, 3.2 and 6.2 minutes on average for $n = 100, 500$ and 1000 respectively. For the censored case the respective computing times are 2.1, 8.3 and 12.6 minutes.

The results for the model without censoring (Table 1) show that our bootstrap test performs very well when $n \geq 500$ with some underrejection for smaller sample size. The test is consistent against both alternatives. Already with a sample size of 500 the test rejects the log-linear and hyperbolic *sin* model almost with certainty.

Table 1: Rejection probabilities, no censoring

	$U \sim Normal$			$U \sim Gumbel$			$U \sim Logistic$		
				$n = 100$					
	10%	5%	1%	10%	5%	1%	10%	5%	1%
Null	7.6	3.7	0.6	6.1	3.6	1.2	5.9	2.3	0.2
Alternative 1	99.8	99.6	98.7	98.7	97.0	89.6	90.5	88.6	82.9
Alternative 2	98.7	94.4	69.0	95.8	87.2	44.3	71.5	45.6	10.6
				$n = 500$					
	10%	5%	1%	10%	5%	1%	10%	5%	1%
Null	10.6	5.7	1.0	9.0	4.4	1.0	8.5	4.3	0.7
Alternative 1	100.0	100.0	100.0	100.0	100.0	100.0	96.5	96.1	95.7
Alternative 2	100.0	100.0	100.0	100.0	100.0	100.0	99.8	97.3	74.7
				$n = 1000$					
	10%	5%	1%	10%	5%	1%	10%	5%	1%
Null	9.7	4.5	0.9	10.4	5.0	1.1	9.4	4.9	1.1
Alternative 1	100.0	100.0	100.0	100.0	100.0	100.0	97.4	97.2	96.4
Alternative 2	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.1

Note: 2000 Monte Carlo simulations, 500 bootstrap replications (parametric bootstrap).

As we can see from Table 2 the finite sample performance of our bootstrap test deteriorates when the dependent variable Y is censored. This is expected because compared to the model with no censoring the rank estimation in the censored model involves additional estimation of the survival function of the censoring threshold C. For example, with a sample of size 100 and censoring rate of 20% we have only about 20 censored observations to estimate this function so the resulting estimator will be quite imprecise. This manifests itself with low power of the test (especially for Alternative 2). However, the power increases fast with the sample size and already with $n = 500$ we reject the alternative models with probability close to one. When it comes to controlling size, even for $n = 1000$ the null rejection probabilities are significantly below the nominal levels which suggests that our test may be conservative in small to medium sized samples. A similar finding was obtained by Subbotin (2007) in his Monte Carlo simulations for the maximum rank correlation estimator of β coefficients in the transformation model.

Overall, our bootstrap test performs reasonably well in small to moderate samples with a tendency to be on the conservative side.

Table 2: Rejection probabilities, random censoring

	$U \sim Normal$			$U \sim Gumbel$			$U \sim Logistic$		
				$n = 100$					
Null	5.2	3.1	0.9	4.3	2.3	0.5	1.9	0.4	0.0
Alternative 1	57.7	36.0	8.5	56.5	35.6	8.8	27.8	12.8	1.5
Alternative 2	26.9	11.5	1.2	16.8	7.7	0.9	7.7	2.6	0.6
				$n = 500$					
	10%	5%	1%	10%	5%	1%	10%	5%	1%
Null	3.4	0.9	0.0	4.4	1.5	0.4	3.3	1.1	0.0
Alternative 1	100.0	99.9	98.7	99.6	97.9	89.1	98.6	96.1	81.0
Alternative 2	99.9	99.9	99.3	99.9	99.7	97.1	98.9	96.2	82.1
				$n = 1000$					
	10%	5%	1%	10%	5%	1%	10%	5%	1%
Null	5.8	2.5	0.4	5.6	2.2	0.1	5.3	2.1	0.4
Alternative 1	100.0	100.0	99.9	99.9	99.7	98.6	99.9	99.8	97.9
Alternative 2	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	99.1

Note: 2000 Monte Carlo simulations, 500 bootstrap replications (nonparametric bootstrap).

3.2 Box-Cox transformation

Due to the high computational burden of implementing the test for the censored Box-Cox model (note that the estimator in Chen (2012) is a two-step estimator and the second step requires minimizing a second order U statistic), we only run a small scale simulation study. We generate data from the log-linear and hyperbolic *sin* model:

$$\log \tilde{Y} = X + U \qquad \text{(Null)}$$

$$\frac{1}{13} \sinh(2 \log(\tilde{Y})) = X + U \qquad \text{(Alternative)}$$

where both X and U are drawn from the standard normal distribution (see right panel of Figure 1). We censor $Y_i = \min\{\tilde{Y}_i, c\}$ where c is chosen to obtain around 20% rate of censoring.

Following the recommendation in Chen (2012) we use uniform weights for Ψ_1 and Ψ_2. In order to estimate the Box-Cox parameter in the second step of his procedure we use grid search. Note that both functions are normalized at $y_0 = 1$. Due to censoring we apply the nonparametric bootstrap procedure.

The results in Table 3 confirm the conclusions from the previous section. Under censoring the nonparametric bootstrap test is conservative in small samples, though rejection probabilities under

Table 3: Censored Box-Cox model, rejection probabilities

	$U \sim Normal$			$U \sim Gumbel$			$U \sim Logistic$		
	$n = 100$								
	10%	5%	1%	10%	5%	1%	10%	5%	1%
Null	0.1	0.1	0.0	0.1	0.0	0.0	0.1	0.0	0.0
Alternative	8.8	2.2	0.0	3.8	0.4	0.0	9.3	1.4	0.0
	$n = 200$								
	10%	5%	1%	10%	5%	1%	10%	5%	1%
Null	0.6	0.4	0.0	0.2	0.0	0.0	0.5	0.2	0.0
Alternative	97.8	92.7	53.6	90.5	72.0	21.4	95.4	85.6	32.6
	$n = 300$								
	10%	5%	1%	10%	5%	1%	10%	5%	1%
Null	2.3	0.4	0.1	1.1	0.4	0.0	2.8	1.0	0.1
Alternative	99.9	99.6	93.5	98.9	97.0	75.4	99.9	99.3	89.6

Note: 1000 Monte Carlo simulations, 500 bootstrap replications (nonparametric bootstrap).

the null get closer to nominal values as the sample size increases. Moreover, the results suggest that the test is consistent.

4 Application to Kennan's strike duration data

In this section we apply our testing procedure in the study of the relation between strike durations and the level of economic activity. Kennan (1985) was the first to empirically investigate this relation using data on strikes involving 1000 or more workers in US manufacturing during 1968-1976. He measured the level of economic activity by an index of industrial production in manufacturing (INDP). Table 4 presents summary statistics.

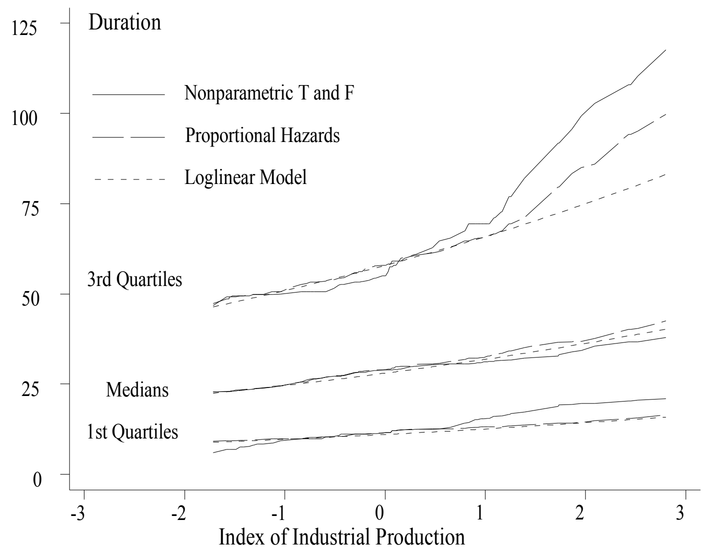
Table 4: Summary statistics, $n = 566$

	Mean	Std. Dev.	Min	Max
strike duration (in days)	43.624	44.666	1	235
INDP	.00604	.04991	-.13996	.08554

Horowitz (2009) re-investigates this question using three models that differ with the parametric assumptions on the transformation function and the distribution F: proportional hazards model (nonparametric Λ, parametric F), loglinear model (parametric Λ, nonparametric F), nonparametric model (both Λ and F nonparametric).

The results of estimating these three models are summarized in Figure 2, which shows estimates

Figure 2: Results of estimating three models of strike duration



Note: This figure comes from Horowitz (2009), Section 6.5. Higher values of the index correspond to lower levels of economic activity.

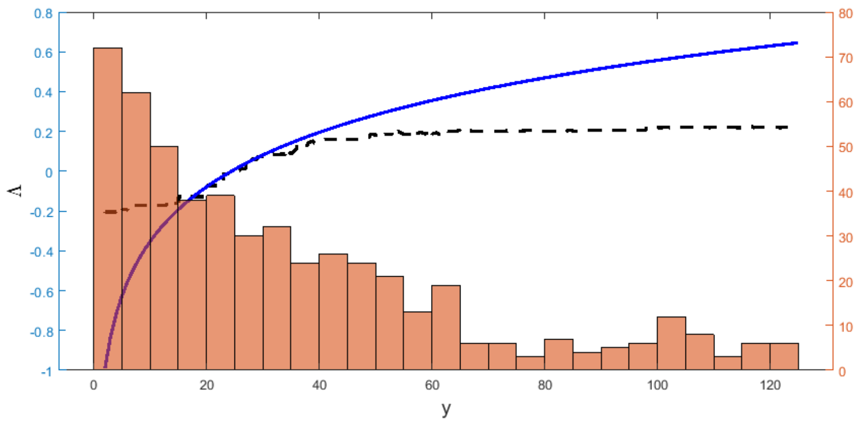
of the conditional first quartile, median, and third quartile of the distribution of strike durations given INDP obtained from each of these models.

For our purpose, it is interesting to compare the loglinear and nonparametric model. These two models differ only with respect to the assumptions on the transformation function, which is exactly the setting that we analyzed above. We notice that the loglinear model and the nonparametric model deliver quite similar predictions for the median strike duration but the results diverge for the first and the third quartile, especially for high values of INDP (i.e. periods of low economic activity). In particular, nonparametric estimates suggest that the distribution of strike durations is more highly skewed to the right than the distribution resulting from the estimation of the loglinear model.

The differences in the estimated parametric and nonparametric transformations are also evident from Figure 3. In particular, the nonparametric curve agrees with log specification around the center of the data (median duration is equal to 28) but diverges further from the median. It is interesting to formally verify if these discrepancies are due merely to the imprecision of the nonparametric estimate in the tails of the data or they signify misspecification of the loglinear model.

For the purpose of our test we set $y_1 = 2$ and $y_2 = 125$ (around 90% of observations on strike

Figure 3: Nonparametric and parametric estimates of the transformation function



Note: Solid line corresponds to the log transformation and dashed line to the nonparametric estimator obtained using the MRC estimator in Chen (2002). Bar plot (right axis) shows the histogram of strike durations.

durations fall in this range) and use Halton sequence of length 100 to evaluate the integral in (2). We run 500 bootstrap replications to obtain the critical value. The test statistic is equal to 43.24 with the bootstrap critical value of 20.35 at the 1% level. We also run a test for $y_1 = 2$ and $y_2 = 61$ (75% of the sample falls in this range) and obtained $T_n = 11.63$ and $c^*_{0.01} = 5.87$. Thus, we reject the loglinear specification and conclude that the differences between the nonparametric and parametric functions in Figure 3 are caused by misspecification of the transformation function rather than being merely a consequence of the estimation error.

5 Discussion

Our test can be embedded into a formalized specification search procedure using ideas in Romano & Wolf (2005). In other words, one can consider multiple parametric null models and run a stepwise multiple testing procedure to choose the correct specification, controlling family-wise error rate at the desired level.

Similarly to testing the form of Λ, one may test the form of the distribution of U using the estimator for F proposed in Ye & Duan (1997) or Horowitz (1996). One can also apply a procedure used in Horowitz (1996) to derive an estimator for F based on CS. Since the estimators of F usually satisfy conditions equivalent to Assumptions 1-5, the same reasoning may be used to derive a CvM test. Such a test may be used to test the form of unobserved heterogeneity (i.e. distribution of V

in (7), F_V) in the MPH model. As pointed out by Heckman & Singer (1984), the estimates of the parameters of the MPH model can be very sensitive to the choice of the parametric form of F_V. Therefore, it may be interesting to see if some parametric specifications are at odds with a nonparametric estimate. Specifically, one may want to test for the presence of unobserved heterogeneity, i.e. test if $V = 0$ in (7). The tests available so far require X to be discrete (usually X distinguishes separate samples), whereas the procedure applied here allows continuously distributed explanatory variables.

Appendix

A Proofs

Let:

$$\mathcal{H} = \{h_{\theta,y}(w_1, w_2, \ldots, w_m) : \theta \in \Theta \subset \mathbb{R}^d, y \in \mathcal{Y} \subset \mathbb{R}_+\}$$

be a family of real-valued functions defined on $\mathcal{W}^m$. We will use the operator notation common in the U-statistics literature. For example, for the case of $m = 2$ we will have $P^0 h = h$, $P^2 h = \int\int h(w_1, w_2) dP(w_1) dP(w_2)$, $P_n h(w_1) = 1/n \sum_{i=1}^n h(w_1, W_i)$ and $P_n^* h(w_1) = 1/n \sum_{i=1}^n h(w_1, W_i^*)$ etc. We say that a symmetric function h is P-canonical if $Ph(w_1, \ldots, w_{m-1}, \cdot) = 0$ for almost all $w_1, \ldots, w_{m-1}$.

Define an U-process:

$$U_n^{(m)} h_{\theta,y} = \frac{(n-m)!}{n!} \sum_{i_1, i_2, \ldots, i_m \text{ distinct}} h_{\theta,y}(W_{i_1}, W_{i_2}, \ldots, W_{i_m})$$

and denote the same process evaluated on a bootstrap sample as $U_n^{*(m)} h_{\theta,y}$.

We will only discuss the model with censoring (i.e. we focus on nonparametric bootstrap) so Y is the censored observation on the dependent variable. Define $\pi(y) = P(Y \geq y)$ and:

$$M(y) = \mathbb{1}\{Y \leq y, \delta = 0\} - \int_0^y \mathbb{1}\{Y \geq u\} d\Lambda_C(u)$$

where Λ_C is the integrated hazard of the censoring variable C. Proofs for the uncensored case (including proofs for parametric bootstrap) follow similar and, in fact, simpler arguments and therefore are omitted.

In order to handle joint randomness in the sample and in bootstrap we can use the following formulation:

$$P_n^* h = \frac{1}{n} \sum_{i=1}^{n} L_{ni} h(W_i)$$

where the bootstrap weights $(L_{n1}, \ldots, L_{nn}) \sim \text{Multinomial}(n, (n^{-1}, \ldots, n^{-1}))$ are defined on the probability space $(\mathcal{L}, \mathcal{C}, P_L)$. We can view $W_1, \ldots, W_n$ as the coordinate projections on the first n coordinates of the canonical probability space $(\mathcal{W}^\infty, \mathcal{A}^\infty, P_W^\infty)$. Thus, for the analysis of joint randomness we can define the product probability space:

$$(\mathcal{W}^\infty, \mathcal{A}^\infty, P_W^\infty) \times (\mathcal{L}, \mathcal{C}, P_L) = (\mathcal{W}^\infty \times \mathcal{L}, \mathcal{A}^\infty \times \mathcal{C}, P_{WL})$$

where $P_{WL} = P_W \times P_L$ since bootstrap weights are independent of the data. Equipped with this formal setup, for any S^* defined on this joint probability space the expectation operator E is understood as:

$$E[S^*] = P_{WL} S^* = P_W P_{L|W} S^*$$

and we say that a real-valued function $f(S^*)$ is of an order $o_p(1)$ when $P_W(P_{L|W}(|f(S^*)| > \epsilon) > \eta) \to 0$ for any $\epsilon, \eta > 0$ as $n \to \infty$. Similarly, $f(S^*)$ is of an order $O_p(1)$ if, for any $\eta > 0$, there exists $0 < K < \infty$ such that $P_W(P_{L|W}(|f(S^*)| > K) > \eta) \to 0$ as $n \to \infty$. We will also frequently use the following stochastic order arithmetic, for a sequence a_n:

$$o_p^*(a_n) + o_p(a_n) = o_p(a_n), \quad O_p^*(a_n) + O_p(a_n) = O_p(a_n)$$

which follows from the Law of Iterated Expectations.[14]

[14] Cheng & Huang (2010) derive such arithmetic for convergence in outer probability.

A.1 Useful lemmas

Lemma 1. (Lo & Singh (1986)) *Let $\bar{G}_0$ be a continuous survival function of the censoring variable and $\bar{G}_n$ and $\bar{G}_n^*$ be Kaplan-Meier estimators of $\bar{G}_0$ on the original and the bootstrap sample, respectively. Then:*

$$\frac{\bar{G}_0(y)-\bar{G}_n(y)}{\bar{G}_0(y)} = P_n \int_0^y \frac{1}{\pi(s)} dM(s) + o_p(n^{-1/2})$$
$$\frac{\bar{G}_0(y)-\bar{G}_n^*(y)}{\bar{G}_0(y)} = P_n^* \int_0^y \frac{1}{\pi(s)} dM(s) + o_p(n^{-1/2})$$

uniformly over $\{y : \pi(y) > c\}$ for some $c > 0$.

Proof. This lemma follows from Theorem 1 in Lo & Singh (1986). They show that uniformly over $\{y : \pi(y) > c\}$:

$$\frac{G_n(y)-G_0(y)}{\bar{G}_0(y)} = P_n\xi(y) + o_p(n^{-1/2}), \qquad \frac{G_n^*(y)-G_n(y)}{\bar{G}_0(y)} = (P_n^* - P_n)\xi(y) + o_p^*(n^{-1/2}),$$

where $\xi(y) = \frac{1}{\pi(Y)}\mathbb{1}\{Y \leq y, \delta = 0\} + \int_0^{\min\{Y,y\}} \frac{1}{\pi(s)^2} d\pi_1(s)$ and $\pi_1(s) = 1 - P(Y \leq s, \delta = 0)$.

But we have $\frac{d\pi_1(s)}{\pi(s)} = d\log \bar{G}_0(s)$ (see equation (7) in Lo & Singh (1986)). Now using the fact that the integrated hazard can be expressed as $\Lambda_C(s) = -\log \bar{G}_0(s)$ and $\frac{1}{\pi(Y)}\mathbb{1}\{Y \leq y, \delta = 0\} = \int_0^y \frac{1}{\pi(s)} d\mathbb{1}\{Y \leq y, \delta = 0\}$ we obtain $\xi(y) = \int_0^y \frac{1}{\pi(s)} dM(y)$. Finally:

$$\frac{\bar{G}_0(y)-\bar{G}_n^*(y)}{\bar{G}_0(y)} = \frac{\bar{G}_0(y)-\bar{G}_n(y)}{\bar{G}_0(y)} + \frac{\bar{G}_n(y)-\bar{G}_n^*(y)}{\bar{G}_0(y)} = P_n^* \int_0^y \frac{1}{\pi(s)} dM(s) + o_p(n^{-1/2})$$

□

The next lemma is similar to Lemma 8 in Subbotin (2007). An important innovation compared to his result is that our proof obviates making additional assumptions about the moments of the kernel function $h \in \mathcal{H}$ (and its envelope) when the arguments of the function are permuted with repetitions. This is possible because decoupling and poissonization arguments akin to those in the proof of Corollary 4.2 in Arcones & Gine (1993) break down the unconditional dependence between arguments of h when evaluated at the bootstrap draws.

Lemma 2. *Let $\mathcal{H}$ be a Euclidean class of P-canonical symmetric functions with envelope H such* *that* $P^m H^{\max\{p,2\}} < \infty$ *for a positive integer* p. *Let* $\mathcal{H}_n$ *be a sequence of subclasses of* $\mathcal{H}$ *with* $\sup_{h\in\mathcal{H}_n} P^m h^2 \to 0$ *as* $n \to \infty$. *Then:*

(a) $\left(P \sup_{h\in\mathcal{H}} \left|U_n^{(m)} h\right|^p\right)^{1/p} = O(n^{-m/2})$

(b) $\left(P \sup_{h\in\mathcal{H}_n} \left|U_n^{(m)} h\right|^p\right)^{1/p} = o(n^{-m/2})$

(c) $\left(P \sup_{h\in\mathcal{H}} \left|U_n^{*(m)} h\right|^p\right)^{1/p} = O(n^{-m/2})$

(d) $\left(P \sup_{h\in\mathcal{H}_n} \left|U_n^{*(m)} h\right|^p\right)^{1/p} = o(n^{-m/2})$

Proof. In order to economize on notation let $\|\cdot\|_{\mathcal{H}} \equiv \sup_{h\in\mathcal{H}} \|\cdot\|$ where $\|\cdot\|$ is the Euclidean norm. We will also write $\lesssim$ for inequality up to a multiplicative constant where the constant does not depend on the sample size n or the sample data (but may depend on p, m and characteristics of $\mathcal{H}$).

Part (a) The result for $p = 1$ follows from Corollary 4(i) in Sherman (1994). Note that by Hoffman-Jørgensen inequality (Corollary 4 in Giné & Zinn (1992)):

$$P\|U_n^{(m)} h\|_{\mathcal{H}}^p \lesssim \left(P\|U_n^{(m)} h\|_{\mathcal{H}}\right)^p + P \max_{i_m} \left\| \frac{(n-m)!}{n!} \sum_{\substack{i_1,\ldots,i_{m-1}: \\ (i_1,\ldots,i_m) \text{ distinct}}} h(W_{i_1}, \ldots, W_{i_m}) \right\|_{\mathcal{H}}^p \tag{13}$$

Thus, it remains to show that the second term is $O(n^{-pm/2})$. We have:

$$\begin{aligned} P \max_{i_m} \left\| \frac{(n-m)!}{n!} \sum_{\substack{i_1,\ldots,i_{m-1}: \\ (i_1,\ldots,i_m) \text{ distinct}}} h(W_{i_1}, \ldots, W_{i_m}) \right\|_{\mathcal{H}}^p &\lesssim P \sum_{i_m} \left\| \frac{(n-m)!}{n(n-1)!} \sum_{\substack{i_1,\ldots,i_{m-1}: \\ (i_1,\ldots,i_m) \text{ distinct}}} h(W_{i_1}, \ldots, W_{i_m}) \right\|_{\mathcal{H}}^p \\ &\lesssim n^{-p+1} E\left[E\left[\left\| U_{n-1}^{(m-1)} h(\cdot, W_{i_m}) \right\|_{\mathcal{H}}^p \middle| W_{i_m} \right] \right] \end{aligned}$$

Partition $W = (W_k, W_{-k})$ where $W_k \in \mathcal{W}^k$ and $W_{-k} \in \mathcal{W}^{m-k}$ and note that fixing $W_{-k} = w_{-k}$ the class of functions $\mathcal{H}_{-k} = \{h(\cdot, w_{-k}) : h \in \mathcal{H}\}$ inherits its properties from $\mathcal{H}$. In particular, for any k and w_{-k} class $\mathcal{H}_{-k}$ is a Euclidean class of P-canonical symmetric functions. Now using (13) for $U_{n-1}^{(m-1)} h(\cdot, W_{i_m})$, Corollary 4(i) in Sherman (1994) and argument in the previous display repeatedly we obtain for $p \geq 2$:

$$P\|U_n^{(m)}h\|_{\mathcal{H}}^p \lesssim O(n^{-pm/2}) + n^{(-p+1)(m-1)}E\left[E\left[\left\|\frac{1}{n-m+1}\sum_{i_1} h(W_{i_1}, W_{i_2}, \ldots, W_{i_m})\right\|_{\mathcal{H}}^p \middle| W_{i_2}, \ldots, W_{i_m}\right]\right]$$

$$\lesssim O(n^{-pm/2}) + n^{(-p+1)m}P^m H^p = O(n^{-pm/2})$$

where H is the envelope of the class $\mathcal{H}$ and the last inequality follows from triangle inequality. This completes the proof of part (a).

Part (b) The result for $p = 1$ follows from Corollary 8 in Sherman (1994). For $p \geq 2$, apply the same iterative argument as in the proof of part (a) repeatedly using $\sup_{h\in\mathcal{H}_n} P^m h^2 \to 0$ and Corollary 8 in Sherman (1994). In the final step we obtain:

$$P\|U_n^{(m)}h\|_{\mathcal{H}_n}^p \lesssim o(n^{-pm/2}) + n^{(-p+1)(m-1)}E\left[E\left[\left\|\frac{1}{n-m+1}\sum_{i_1} h(W_{i_1}, W_{i_2}, \ldots, W_{i_m})\right\|_{\mathcal{H}_n}^p \middle| W_{i_2}, \ldots, W_{i_m}\right]\right]$$

$$\lesssim o(n^{-pm/2}) + n^{(-p+1)(m-1)}E\left[E\left[\|(P_n - P)h(\cdot, W_{i_2}, \ldots, W_{i_m})\|_{\mathcal{H}_n}^p \middle| W_{i_2}, \ldots, W_{i_m}\right]\right]$$

$$\lesssim o(n^{-pm/2}) + o(n^{-pm/2})n^{(1-p/2)(m-1)} = o(n^{-pm/2})$$

where the second inequality is a result of h being P-canonical and the third follows from asymptotic equicontinuity of the empirical process $(P_n - P)h$ (using $\sup_{h\in\mathcal{H}_n} P^m h^2 \to 0$).

Part (c) Proof of this part follows from applying decoupling and poissonization techniques used in Gine & Zinn (1990) and Arcones & Gine (1993). First by Hoeffding decomposition:

$$U_n^{*(m)}h = \sum_{k=0}^{m}\binom{m}{k}U_n^{*(k)}\left(\pi_{k,m}^{P_n}h\right) \tag{14}$$

where $\pi_{k,m}^{P_n}h(w_1, \ldots, w_k) = (\delta_{w_1} - P_n)\ldots(\delta_{w_k} - P_n)P_n^{m-k}h$ and $\delta_{w_1}h = h(w_1, \cdot)$.

We will show that $P\|U_n^{*(k)}(\pi_{k,m}^{P_n}h)\|_{\mathcal{H}}^p = O(n^{-pm/2})$ for each $k = 0, \ldots, m$. Note that under our assumptions $\pi_{k,m}^{P_n}h(w_1, \ldots, w_k)$ are P_n-canonical symmetric functions. Let $\{W_i^{*(k)} : i = 1, \ldots, n\}_{k=1}^m$ be i.i.d. copies of $\{W_i^* : i = 1, \ldots, n\}$ and let $\{\epsilon_i : i = 1, \ldots, n\}$ denote a sequence of Rademacher random variables independent of W_i^*'s. Similarly, define $\{\epsilon_i^{(k)} : i = 1, \ldots, n\}_{k=1}^m$ to be independent copies of $\{\epsilon_i : i = 1, \ldots, n\}$. We have:

$$P\left\|U_n^{*(k)}\left(\pi_{k,m}^{P_n} h\right)\right\|_{\mathcal{H}}^p = P\left\|\frac{(n-k)!}{n!}\sum_{i_1,\ldots,i_k \text{ distinct}} \pi_{k,m}^{P_n} h(W_{i_1}^*, \ldots, W_{i_k}^*)\right\|_{\mathcal{H}}^p$$
$$\lesssim P\left\|\frac{(n-k)!}{n!}\sum_{i_1,\ldots,i_k \text{ distinct}} \epsilon_{i_1}^{(1)} \ldots \epsilon_{i_k}^{(k)} \pi_{k,m}^{P_n} h(W_{i_1}^{*(1)}, \ldots, W_{i_k}^{*(k)})\right\|_{\mathcal{H}}^p$$
$$\lesssim P\left\|\frac{(n-k)!}{n!}\sum_{i_1,\ldots,i_k \text{ distinct}} \epsilon_{i_1}^{(1)} \ldots \epsilon_{i_k}^{(k)} P_n^{m-k} h(W_{i_1}^{*(1)}, \ldots, W_{i_k}^{*(k)})\right\|_{\mathcal{H}}^p = (\star)$$

where the first inequality follows by decoupling (see Proposition 2.1 in Arcones & Gine (1993)) and the second by repeated application of Jensen's inequality (argument here follows from the fact that by Jensen's inequality $E\|X\|_{\mathcal{H}}^p \leq E\|X+Y\|_{\mathcal{H}}^p$ when $E[Y|X]=0$, for example when $k=m=2$ and letting $\widetilde{W}_i^{*(1)}$ denote an independent copy of $W_i^{*(1)}$ and $\mathcal{EW} = \{\epsilon_i^{(1)}, \epsilon_i^{(2)}, W_i^{*(1)}, W_i^{*(2)}\}_{i=1}^n$ we have:

$$P\left\|\sum_{i\neq j} \epsilon_i^{(1)} \epsilon_j^{(2)} \pi_{2,2}^{P_n} h(W_i^{*(1)}, W_j^{*(2)})\right\|_{\mathcal{H}}^p =$$
$$= P\left\|\sum_{i\neq j} \epsilon_i^{(1)} \epsilon_j^{(2)} (\delta_{W_i^{*(1)}} - P_n)(\delta_{W_j^{*(2)}} - P_n)h - E_{P_n}\left[\sum_{i\neq j} \epsilon_i^{(1)} \epsilon_j^{(2)} (\delta_{\widetilde{W}_i^{*(1)}} - P_n)(\delta_{W_j^{*(2)}} - P_n)h \middle| \mathcal{EW}\right]\right\|_{\mathcal{H}}^p$$
$$\lesssim P\left\|\sum_{i\neq j} \epsilon_i^{(1)} \epsilon_j^{(2)} \delta_{W_i^{*(1)}} (\delta_{W_j^{*(2)}} - P_n)h - \sum_{i\neq j} \epsilon_i^{(1)} \epsilon_j^{(2)} \delta_{\widetilde{W}_i^{*(1)}} (\delta_{W_j^{*(2)}} - P_n)h\right\|_{\mathcal{H}}^p$$
$$\lesssim P\left\|\sum_{i\neq j} \epsilon_i^{(1)} \epsilon_j^{(2)} \delta_{W_i^{*(1)}} (\delta_{W_j^{*(2)}} - P_n)h\right\|_{\mathcal{H}}^p \lesssim \ldots \lesssim P\left\|\sum_{i\neq j} \epsilon_i^{(1)} \epsilon_j^{(2)} h(W_i^{*(1)}, W_j^{*(2)})\right\|_{\mathcal{H}}^p$$

where E_{P_n} denotes expectation with respect to the empirical measure P_n).

Introduce $Z_{i_1}^* = \sum_{i_2,\ldots,i_k \neq i_1 \text{ distinct}} \epsilon_{i_2}^{(2)} \ldots \epsilon_{i_k}^{(k)} P_n^{m-k} h(W_{i_1}^{*(1)}, \ldots, W_{i_k}^{*(k)})$. Conditional on $\{\epsilon_{i_2}^{(2)}, \ldots, \epsilon_{i_k}^{(k)}, W_{i_2}^{*(2)}, \ldots, W_{i_k}^{*(k)}\}$, $Z_{i_1}^*$'s are bootstrap draws from the sample $\{Z_i : i = 1, \ldots, n\}$ where $Z_{i_1} = \sum_{i_2,\ldots,i_k \neq i_1 \text{ distinct}} \epsilon_{i_2}^{(2)} \ldots \epsilon_{i_k}^{(k)} P_n^{m-k} h(W_{i_1}, W_{i_2}^{*(2)}, \ldots, W_{i_k}^{*(k)})$.

Let $\{\tilde{N}_i^{(k)} : i = 1, \ldots, n\}_{k=1}^m$ denote independent copies of a sequence of differences between two independent Poisson random variables with parameter $1/2$. Now applying Proposition 2.2 in Gine

& Zinn (1990) we get:

$$
\begin{aligned}
(\star) &= P\left\| \frac{(n-k)!}{n!} \sum_{i_1} \epsilon_{i_1}^{(1)} Z_{i_1}^* \right\|_{\mathcal{H}}^p \lesssim P\left\| \frac{(n-k)!}{n!} \sum_{i_1} \tilde{N}_{i_1}^{(1)} Z_{i_1} \right\|_{\mathcal{H}}^p \lesssim P\left\| \frac{(n-k)!}{n!} \sum_{i_1} N_{i_1}^{(1)} Z_{i_1} \right\|_{\mathcal{H}}^p \\
&= P\left\| \frac{(n-k)!}{n!} \sum_{i_1, i_2 \text{ distinct}} N_{i_1}^{(1)} \epsilon_{i_2}^{(2)} Z_{i_2}^* \right\|_{\mathcal{H}}^p
\end{aligned}
$$

where N_{i_1} is a Poisson random variable with parameter $1/2$ and

$$
Z_{i_2}^* = \sum_{i_3, \ldots, i_k \neq i_1, i_2, \text{ distinct}} \epsilon_{i_3}^{(3)} \ldots \epsilon_{i_k}^{(k)} P_n^{m-k} h(W_{i_1}, W_{i_2}^{*(2)}, \ldots, W_{i_k}^{*(k)})
$$

The second inequality follows from triangle inequality. Iterating in this fashion and repeatedly using Proposition 2.2 in Gine & Zinn (1990) (additionally conditioning on previously introduced Poisson variables) we obtain:

$$
\begin{aligned}
(\star) &\lesssim P\left\| \frac{(n-k)!}{n!} \sum_{i_1, \ldots, i_k \text{ distinct}} N_{i_1}^{(1)} \ldots N_{i_k}^{(k)} P_n^{m-k} h(W_{i_1}, \ldots, W_{i_k}) \right\|_{\mathcal{H}}^p \\
&= P\left\| \frac{(n-k)!}{n!} \sum_{i_1, \ldots, i_k \text{ distinct}} N_{i_1}^{(1)} \ldots N_{i_k}^{(k)} E_{P_n}\left[\frac{1}{n-k} \sum_{i_{k+1} \neq i_1, \ldots, i_k} P_n^{m-k-1} h(W_{i_1}, \ldots, W_{i_k}, W_{i_{k+1}}^*) \middle| W_{i_1}, \ldots, W_{i_k} \right] \right\|_{\mathcal{H}}^p \\
&\lesssim P\left\| \frac{(n-k)!}{n!(n-k)} \sum_{i_1, \ldots, i_{k+1} \text{ distinct}} N_{i_1}^{(1)} \ldots N_{i_k}^{(k)} P_n^{m-k-1} h(W_{i_1}, \ldots, W_{i_k}, W_{i_{k+1}}) \right\|_{\mathcal{H}}^p \lesssim \cdots \\
&\lesssim P\left\| \frac{(n-m)!}{n!} \sum_{i_1, \ldots, i_m \text{ distinct}} N_{i_1}^{(1)} \ldots N_{i_k}^{(k)} h(W_{i_1}, \ldots, W_{i_m}) \right\|_{\mathcal{H}}^p \\
&\lesssim P\left\| \frac{(n-m)!}{n!} \sum_{i_1, \ldots, i_m \text{ distinct}} N_{i_1}^{(1)} \ldots N_{i_m}^{(m)} h(W_{i_1}, \ldots, W_{i_m}) \right\|_{\mathcal{H}}^p
\end{aligned}
$$

where E_{P_n} denotes expectation with respect to the empirical measure P_n, second inequality follows from Jensens's inequality, fourth line comes from iterating this argument and final inequality follows again from Jensen's inequality using the fact that $E[N_{i_{k+1}}^{(k+1)} \ldots N_{i_k}^{(m)}] > 0$.

Now define $\tilde{h}((N_{i_1}^{(1)}, W_{i_1}), \ldots, (N_{i_m}^{(m)}, W_{i_m})) = N_{i_1}^{(1)} \ldots N_{i_m}^{(m)} h(W_{i_1}, \ldots, W_{i_m})$, let $\tilde{P}$ denote the distribution of $\{W_i, N_i^{(1)}, \ldots, N_i^{(m)}\}$ and note that the class of functions $\tilde{\mathcal{H}}$ inherits its properties from $\mathcal{H}$, in particular it is a Euclidean class of $\tilde{P}$-canonical symmetric functions with envelope $\tilde{H}$ satisfying $\tilde{P}^m \tilde{H}^{max\{p,2\}} < \infty$ for $p \geq 1$. Hence, our derivation and part (a) imply that for any

$k \leq m$:

$$P\|U_n^{*(k)}\left(\pi_{k,m}^{P_n} h\right)\|_{\mathcal{H}}^p \lesssim P\|U_n^{(m)}\tilde{h}\|_{\tilde{\mathcal{H}}}^p = O(n^{-pm/2}) \tag{15}$$

and the final result follows from (14).

Part (d) Note that $\sup_{h\in\mathcal{H}_n} P^m h^2 \to 0$ implies $\sup_{\tilde{h}\in\tilde{\mathcal{H}}_n} \tilde{P}^m\tilde{h}^2 \to 0$ as N_i's are independent of W_i's. So the result follows from (15) and part (b).

□

Lemma 3. *Let $h_{\theta_0,y} = 0$ for all $y \in \mathcal{Y}$ and define $\tau_{\theta,y}(w) = P^{m-1}h_{\theta,y}(w)$. If $|h_{\theta,y}(w_1,\ldots,w_m)| < H$ for some $0 < H < \infty$ and:*

(a) $\Theta \subset \mathbb{R}^{d_\theta}$ and $\mathcal{Y} \subset \mathbb{R}^{d_y}$ are compact sets, $P^m h_{\theta,y}$ is continuous on Θ for every $y \in \mathcal{Y}$,

(b) $\mathcal{H}$ is a Euclidean class of symmetric functions,

(c) there is an open neighborhood $\mathcal{N} \subset \Theta$ of θ_0 such that:

(i) all mixed partial derivatives of $\tau_{\theta,y}(w)$ with respect to θ of orders 1 and 2 exist on $\mathcal{N}$ for all $y \in \mathcal{Y}$,

(ii) there is a square P-integrable function $K(w)$ such that for all w, $y, y' \in \mathcal{Y}$ and all θ in $\mathcal{N}$:

$$\|vec(\partial^2\tau_{\theta,y}(w)) - vec(\partial^2\tau_{\theta_0,y'}(w))\| \leq K(w)\sqrt{\|\theta-\theta_0\|^2 + \|y-y'\|^2}$$

where $\partial^2\tau$ is the Hessian matrix of τ with respect to θ,

(iii) the gradient of $\tau_{\theta,y}$ with respect to θ at θ_0, $\partial\tau_{\theta_0,y}(w)$, has finite variance relative to P for all $y \in \mathcal{Y}$ and $P\partial\tau_{\theta_0,y} = 0$,

(iv) the elements of the matrix $A(y) = -P[\partial^2\tau_{\theta_0,y}]$ are finite for all $y \in \mathcal{Y}$

(d) as $\theta \to \theta_0$, $P^m h_{\theta,y}^2 \to 0$ for all $y \in \mathcal{Y}$,

then

$$P \sup_{\theta\in\Theta,y\in\mathcal{Y}} |U_n^{*(m)}h_{\theta,y} - P^m h_{\theta,y}| \to 0 \tag{16}$$

and uniformly over $y \in \mathcal{Y}$:

$$U_n^{(m)} h_{\theta,y} = (\theta - \theta_0)' m P_n \partial \tau_{\theta_0,y} - \frac{1}{2}(\theta - \theta_0)' A(y)(\theta - \theta_0) + o_p(\|\theta - \theta_0\|^2) + o_p(n^{-1}) \tag{17}$$

$$U_n^{*(m)} h_{\theta,y} = (\theta - \theta_0)' m P_n^* \partial \tau_{\theta_0,y} - \frac{1}{2}(\theta - \theta_0)' A(y)(\theta - \theta_0) + o_p(\|\theta - \theta_0\|^2) + o_p(n^{-1}) \tag{18}$$

as $\theta \to \theta_0$.

Proof. This result follows from Lemma 2 and arguments leading to Theorem 2 in Subbotin (2007). The only difference is that the function h is indexed by y in addition to θ. Also note that we do not need invertibility of A here (his Assumption 3(iv)). For completeness we give details of the proof of (18) ((17) follows by similar arguments).

Use the following Hoeffding decomposition for the bootstrapped U-statistic (see Subbotin (2007) for details):

$$U_n^{*(m)} h_{\theta,y} = (\theta - \theta_0)' m P_n^* \partial \tau_{\theta_0,y} - \frac{1}{2}(\theta - \theta_0)' A(y)(\theta - \theta_0) + \hat{\zeta}_{\theta,y}$$

where

$$\hat{\zeta}_{\theta,y} = P_n^* R_{\theta,y} + \sum_{k=2}^{m} \binom{m}{k} U_n^{*(k)} \pi_{k,m}^{P} h_{\theta,y}(w_1, \ldots, w_k)$$

$$\delta_{w_k} h_{\theta,y}(\cdot) = h_{\theta,y}(\cdot, w_k, \cdot)$$

$$R_{\theta,y}(w) = [P^m h_{\theta,y} + m \pi_{1,m}^{P} \tau_{\theta,y}](w) - m(\theta - \theta_0)' \partial \tau_{\theta_0,y}(w) + \frac{1}{2}(\theta - \theta_0)' A(y)(\theta - \theta_0).$$

and $\pi_{k,m}^{P}$ is defined analogously to $\pi_{k,m}^{P_n}$ above.

Condition (c) and second order Taylor expansion around θ_0 imply:

$$|P_n^* R_{\theta,y}| \le m \|(P_n^* - P)\partial^2 \tau_{\theta_0,y}\| \|\theta - \theta_0\|^2 + m(PK + P_n^* K)\|\theta - \theta_0\|^3$$

in the neighborhood of θ_0.

First we will show that

$$\sup_{y, \|\theta - \theta_0\| \le \delta_n} \frac{|P_n^* R_{\theta,y}|}{\|\theta - \theta_0\|^2} = o_p(1) \tag{19}$$

for $\delta_n \to 0$. Note that condition (c) implies that $PK + P_n^* K = O_p(1)$ (by Theorem 2.1 in Bickel & Freedman (1981)) and that $\partial^2 \mathcal{T}_0 = \{vec(\partial^2 \tau_{\theta_0,y}(w)) : y \in \mathcal{Y} \subset \mathbb{R}^{d_y}\}$ is a Euclidean class of functions (by Lemma 2.13 in Pakes & Pollard (1989)). Thus, by the uniform law of large numbers and bootstrap uniform law of large numbers (Theorem 3.5 in Gine & Zinn (1990)) we have $\|(P_n - P)\partial^2 \tau_{\theta_0,y}\| = o_p(1)$ and $\|(P_n^* - P_n)\partial^2 \tau_{\theta_0,y}\| = o_p^*(1)$ uniformly over y, which implies $\sup_y \|(P_n^* - P)\partial^2 \tau_{\theta_0,y}\| = o_p(1)$ and (19) follows.

Using conditions (b), (d) and Lemma 2 we get:[15]

$$P \sup_{y, \|\theta-\theta_0\| \le \delta_n} \left| \sum_{k=2}^{m} \binom{m}{k} U_n^{*(k)} \pi_{k,m}^P h_{\theta,y}(w_1, \ldots, w_k) \right| = o(n^{-1})$$

which, together with (19), implies

$$\sup_{y, \|\theta-\theta_0\| \le \delta_n} |\hat{\zeta}_{\theta,y}| = o_p(\|\theta - \theta_0\|^2) + o_p(n^{-1}).$$

This concludes the proof of (18). □

A.2 Proof of Theorem 1

Assumptions 1-2 imply that $\hat{\gamma} - \gamma = O_p(n^{-1/2})$, $\hat{\beta}_1 - \beta_1 = O_p(n^{-1/2})$ and $\Lambda_n(y) - \Lambda_0(y) = O_p(n^{-1/2})$ uniformly over $y \in [y_1, y_2]$. Thus:

$$T_n = \int_{y_1}^{y_2} [S_{n1}(y) + S_{n2}(y) + S_{n3}(y) + S_{n4}(y)]^2 dy + o_p(1)$$

uniformly over y, where: $S_{n1}(y) = \sqrt{n}\beta_1(\Lambda_n(y) - \Lambda_0(y))w(y)$, $S_{n2}(y) = -\sqrt{n}(\Lambda(y, \hat{\gamma}) - \Lambda(y, \gamma))w(y)$, $S_{n3}(y) = \sqrt{n}\Lambda_0(y)(\hat{\beta}_1 - \beta_1)w(y)$, $S_{n4}(y) = \sqrt{n}(\beta_1 \Lambda_0(y) - \Lambda(y, \gamma))w(y)$.

Under the null we have $S_{n4}(y) = 0$ and by Assumption 2(c):

$$\sqrt{n}(\Lambda(y, \hat{\gamma}) - \Lambda(y, \gamma)) = -\sqrt{n} \frac{\partial \Lambda(y, \gamma)'}{\partial \gamma} P_n \Omega_\gamma + o_p(1)$$

[15] Note that Jensen's inequality implies $P^m(\pi_{k,m}^P h_{\theta,y})^2 \lesssim P^m h_{\theta,y}^2$. Thus, the second condition in Lemma 2 (with $\mathcal{H}_n = \{\pi_{k,m}^P h_{\theta,y} : y \in \mathcal{Y}, \|\theta - \theta_0\| \le \delta_n\}$) follows from condition (d).

uniformly over y. Hence, using Assumptions 1-3 we get:

$$T_n = \int_{y_1}^{y_2} B_n(y)^2 dy + o_p(1)$$

and the statement of the theorem follows from extended continuous mapping theorem (Theorem 1.11.1 in Van der Vaart & Wellner (1996)) and the results in Durbin & Knott (1972), Durbin et al. (1975).

A.3 Proof of Theorem 2

We have:

$$T_n^* = \int_{y_1}^{y_2} [S_{n1}^*(y) + S_{n2}^*(y) + S_{n3}^*(y)]^2 dy$$

uniformly over y, where: $S_{n1}^*(y) = \sqrt{n}\hat{\beta}_1(\Lambda_n^*(y) - \Lambda_n(y))w(y)$, $S_{n2}^*(y) = -\sqrt{n}(\Lambda(y, \hat{\gamma}^*) - \Lambda(y, \hat{\gamma}))w(y)$, $S_{n3}^*(y) = \sqrt{n}\Lambda_n^*(y)(\hat{\beta}_1^* - \hat{\beta}_1)w(y)$.

We need to obtain a bootstrap linear approximation to $\sqrt{n}(\Lambda_n^*(y) - \Lambda_n(y))$. Let $\theta = (b, \Lambda)$ where $b \in \Theta_\beta$ and $\Lambda \in \Theta_\Lambda$. Let $\Gamma^*(y, G, \Lambda, b) = U_n^*[r(w_1, w_2, G, y, \Lambda, b) + r(w_2, w_1, G, y, \Lambda, b)]$ denote the symmetrized bootstrap rank objective function recentered at the true value Λ_0 and note that Λ_n^* is its arg max. Similarly, let $\Gamma(y, G, \Lambda, b) = P^2[r(W_1, W_2, G, y, \Lambda, b) + r(W_2, W_1, G, y, \Lambda, b)]$. Define:

$$\begin{aligned}
h_{\theta,y}^1(w_1, w_2) &= \mathbb{1}\{y^1 \geq y\}(\mathbb{1}\{x^1 b - x^2 b \geq \Lambda\} - \mathbb{1}\{x^1 b - x^2 b \geq \Lambda_0\}) \\
&\quad + \mathbb{1}\{y^2 \geq y\}(\mathbb{1}\{x^2 b - x^1 b \geq \Lambda\} - \mathbb{1}\{x^2 b - x^1 b \geq \Lambda_0\}) \\
h_{\theta,y}^2(w_1, w_2) &= \mathbb{1}\{y^1 \geq y_0\}(\mathbb{1}\{x^1 b - x^2 b \geq \Lambda\} - \mathbb{1}\{x^1 b - x^2 b \geq \Lambda_0\}) \\
&\quad + \mathbb{1}\{y^2 \geq y_0\}(\mathbb{1}\{x^2 b - x^1 b \geq \Lambda\} - \mathbb{1}\{x^2 b - x^1 b \geq \Lambda_0\}).
\end{aligned}$$

We have:

$$\begin{aligned}
\Gamma^*(y, G^*, \Lambda, b) &= \Gamma^*(y, G_0, \Lambda, b) + \Gamma^*(y, G^*, \Lambda, b) - \Gamma^*(y, G_0, \Lambda, b) \\
&= \frac{1}{\bar{G}_0(y)} U_n^* h_{\theta,y}^1 - \frac{1}{\bar{G}_0(y_0)} U_n^* h_{\theta,y}^2 + \frac{\bar{G}_0(y) - \bar{G}^*(y)}{\bar{G}^*(y)\bar{G}_0(y)} U_n^* h_{\theta,y}^1 - \frac{\bar{G}_0(y_0) - \bar{G}^*(y_0)}{\bar{G}^*(y_0)\bar{G}_0(y_0)} U_n^* h_{\theta,y}^2. \qquad (20)
\end{aligned}$$

Define $\tau^l_{\theta,y}(w) = Ph^l_{\theta,y}(w)$ and $A^l(y) = -P[\partial^2 \tau^l_{\theta,y}]$ for $l = 1, 2$. We will use Lemma 3 to show that

$$P \sup_{\theta\in\Theta, y\in\mathcal{Y}} |U^*_n h^l_{\theta,y} - P^2 h^l_{\theta,y}| \to 0 \tag{21}$$

and uniformly over $y \in [y_1, y_2]$:

$$U_n h^l_{\theta,y} = (\theta-\theta_0)' 2P_n \partial\tau^l_{\theta_0,y} - \frac{1}{2}(\theta-\theta_0)' A^l(y)(\theta-\theta_0) + o_p(\|\theta-\theta_0\|^2) + o_p(n^{-1}) \tag{22}$$

$$U^*_n h^l_{\theta,y} = (\theta-\theta_0)' 2P^*_n \partial\tau^l_{\theta_0,y} - \frac{1}{2}(\theta-\theta_0)' A^l(y)(\theta-\theta_0) + o_p(\|\theta-\theta_0\|^2) + o_p(n^{-1}) \tag{23}$$

for $l = 1, 2$ as $\theta \to \theta_0$.

Let us verify conditions of Lemma 3. Condition (a) is implied by Assumptions 4(b),(d),(e). Chen (2002) showed that the classes of functions

$$\mathcal{H}^l = \{h^l_{\theta,y}(w_1, w_2) : \theta \in \Theta \subset \mathbb{R}^d, y \in \mathcal{Y} \subset \mathbb{R}_+\} \qquad l = 1, 2$$

are Euclidean for the envelope $H = 2$, thus condition (b) is satisfied. Condition (c) is implied by Assumption 4(e). Finally, continuity of the distribution of U and X_1 imply condition (d).

Now note that Lemma 1 and Assumption 5 imply that $\frac{\bar{G}_0(y)-\bar{G}^*_n(y)}{\bar{G}_0(y)} = o_p(1)$ and $b^*_n \to^p \beta_0$. Combining this, equation (20), the result in (21) and using Assumption 4(e) we obtain:

$$\Gamma^*(y, G^*, \Lambda, b^*_n) = \Gamma(y, G_0, \Lambda, \beta_0) + o_p(1)$$

uniformly over $y \in \mathcal{Y}$ and $\Lambda \in \Theta_\Lambda$. Chen (2002) showed that Λ_0 is the unique maximizer of the expression on the right, which implies consistency of $\Lambda^*_n(y)$ for $\Lambda_0(y)$. Now monotonicity of $\Lambda^*_n(y)$ implies uniform consistency, i.e. $\sup_y |\Lambda^*_n(y) - \Lambda_0(y)| = o_p(1)$, by the same argument as in the proof of Theorem 1 in Chen (2002).

Note that $\frac{\partial\tau^l_{\theta,y}}{\partial b}\big|_{\Lambda=\Lambda_0} = 0$ and $P\frac{\partial^2\tau^l_{\theta,y}}{\partial b^2}\big|_{\Lambda=\Lambda_0} = 0$. Let $V^l_{\Lambda b}(y) = -P\frac{\partial^2\tau^l_{\theta,y}}{\partial\Lambda\partial b}\big|_{\theta=\theta_0}$ and $V^l(y) = -P\frac{\partial^2\tau^l_{\theta,y}}{\partial\Lambda^2}\big|_{\theta=\theta_0}$. Then (23) becomes:

$$U^*_n h^l_{\theta,y} = (\Lambda-\Lambda_0)2P^*_n \frac{\partial\tau^l_{\theta_0,y}}{\partial\Lambda} - (\Lambda-\Lambda_0)V^l_{\Lambda b}(y)'(b-\beta_0) - \frac{1}{2}(\Lambda-\Lambda_0)^2 V^l(y) + o_p((\Lambda-\Lambda_0)^2) + o_p(n^{-1}) \tag{24}$$

as $\Lambda \to \Lambda_0$ and $b \to \beta_0$.

Chen (2002) shows that under our assumptions the class of functions $\partial\mathcal{T}_0 = \{\partial\tau_{\theta_0,y}(w) : y \in \mathcal{Y} \subset \mathbb{R}_+\}$ is Euclidean with a square integrable envelope. Similar argument shows that the same property holds for $\partial^2\mathcal{T}_0 = \{vec(\partial^2\tau_{\theta_0,y}(w)) : y \in \mathcal{Y} \subset \mathbb{R}_+\}$ (see also proof of Lemma 3). Thus, Theorem 3.5 in Gine & Zinn (1990) gives: $\sup_y \|(P_n^* - P_n)\partial\tau_{\theta_0,y}\| = O_p^*(n^{-1/2})$ and $\sup_y \|(P_n^* - P_n)\partial^2\tau_{\theta_0,y}\| = O_p^*(n^{-1/2})$ and similarly $\sup_y \|(P_n - P)\partial\tau_{\theta_0,y}\| = O_p(n^{-1/2})$ and $\sup_y \|(P_n - P)\partial^2\tau_{\theta_0,y}\| = O_p(n^{-1/2})$.

This and Lemma 1 imply that the third and the fourth term in (20) can be written as:

$$(\Lambda - \Lambda_0)\left(\frac{2}{\bar{G}_0(y)}P\frac{\partial\tau^1_{\theta_0,y}}{\partial\Lambda}P_n^*\int_0^y \frac{1}{\pi}dM - \frac{2}{\bar{G}_0(y_0)}P\frac{\partial\tau^2_{\theta_0,y}}{\partial\Lambda}P_n^*\int_0^{y_0} \frac{1}{\pi}dM\right)$$
$$+ o_p((\Lambda - \Lambda_0)^2) + o_p((\Lambda - \Lambda_0)/\sqrt{n}) + o_p(n^{-1}) \qquad (25)$$

uniformly over y.

Note that $V(y) = \frac{V^1(y)}{\bar{G}_0(y)} - \frac{V^2(y)}{\bar{G}_0(y_0)}$ and $\frac{\partial\tau(W,y,\Lambda_0)}{\partial\Lambda} = \frac{1}{\bar{G}_0(y)}\frac{\partial\tau^1_{\theta_0,y}}{\partial\Lambda} - \frac{1}{\bar{G}_0(y_0)}\frac{\partial\tau^2_{\theta_0,y}}{\partial\Lambda}$. Define $V_{\Lambda b} = \frac{V^1_{\Lambda b}}{\bar{G}_0(y)} - \frac{V^2_{\Lambda b}}{\bar{G}_0(y_0)}$. Thus, substituting (24) and (25) into (20) and using Assumption 5 we obtain:

$$\Gamma^*(y, G^*, \Lambda, b_n^*) = (\Lambda - \Lambda_0)P_n^*\Omega_{\Lambda,y} - \frac{1}{2}(\Lambda - \Lambda_0)^2 V(y) + o_p((\Lambda - \Lambda_0)^2) + o_p((\Lambda - \Lambda_0)/\sqrt{n}) + o_p(n^{-1})$$

where

$$\Omega_{\Lambda,y} = 2\frac{\partial\tau(W,y,\Lambda_0)}{\partial\Lambda} + \frac{2}{\bar{G}_0(y)}\int_0^y \frac{1}{\pi}dM\left(P\frac{\partial\tau^1_{\theta_0,y}}{\partial\Lambda}\right) - \frac{2}{\bar{G}_0(y_0)}\int_0^{y_0} \frac{1}{\pi}dM\left(P\frac{\partial\tau^2_{\theta_0,y}}{\partial\Lambda}\right) - V_{\Lambda b}(y)'\Omega^{NP}$$

uniformly over y. Now using $\sup_y |\Lambda_n^*(y) \to \Lambda_0(y)| \to 0$ one can proceed as in Sherman (1993) to show that:

$$\sqrt{n}(\Lambda_n^*(y) - \Lambda_0(y)) = V(y)^{-1}P_n^*\Omega_{\Lambda,y} + o_p(1)$$

uniformly over y. From Chen (2002) and Jochmans (2012):

$$\sqrt{n}(\Lambda_n(y) - \Lambda_0(y)) = V(y)^{-1}P_n\Omega_{\Lambda,y} + o_p(1)$$

and the class of functions $\mathcal{J} = \{J(\cdot, y) = V(y)^{-1}\Omega_{\Lambda,y}(\cdot)\}$ is Euclidean with square integrable envelope. Now:

$$\sqrt{n}(\Lambda_n^*(y) - \Lambda_n(y)) = V(y)^{-1}(P_n^* - P_n)\Omega_{\Lambda,y} + o_p(1)$$

and by Theorem 3.5 in Gine & Zinn (1990) we have $\sup_y |\Lambda_n^*(y) - \Lambda_n(y)| = O_p(n^{-1/2})$, which together with Assumption 2(d) yields:

$$S_{n1}^*(y) = \sqrt{n}\beta_1 V(y)^{-1}(P_n^* - P_n)\Omega_{\Lambda,y}w(y) + o_p(1).$$

uniformly over y. Further, Assumptions 2(b)-(d), Assumption 5 and Theorem 2.2 in Bickel & Freedman (1981) imply:

$$S_{n2}^*(y) = -\sqrt{n}\frac{\partial\Lambda(y,\gamma)}{\partial\gamma}' (P_n^* - P_n)\Omega_\gamma w(y) + o_p(1)$$
$$S_{n3}^*(y) = \sqrt{n}\Lambda_0(y)(P_n^* - P_n)\Omega_1 w(y) + o_p(1)$$

uniformly over y. Denote $\mathcal{B}_n^*(y) = \sqrt{n}(P_n^* - P_n)[\beta_1 V(y)^{-1}\Omega_{\Lambda,y} - \frac{\partial\Lambda(y,\gamma)}{\partial\gamma}\Omega_\gamma + \Lambda_0(y)\Omega_1]w(y)$. Now note that functions Ω_γ and Ω_1 are not indexed by y and $w(y)$, $\frac{\partial\Lambda(y,\gamma)}{\partial\gamma}w(y)$ and $\Lambda_0(y)w(y)$ are constant for fixed y. Thus, by Lemma 2.14 in Pakes & Pollard (1989), Theorem 3.5 in Gine & Zinn (1990) and the extended continuous mapping theorem (Theorem 1.11.1 in Van der Vaart & Wellner (1996)) we have that $\int_{y_1}^{y_2} \mathcal{B}_n^{*2}(y)dy$ converges weakly to $\int_{y_1}^{y_2} \mathcal{B}^2(y)dy$ in conditional probability. Additionally, by continuity of the distribution of $\int_{y_1}^{y_2} \mathcal{B}^2(y)dy$ and monotonicity of CDFs this implies:

$$\sup_{t\geq 0}\left|P_n^*\mathbb{1}\left\{\int_{y_1}^{y_2} \mathcal{B}_n^{*2}(y)dy \leq t\right\} - P\mathbb{1}\left\{\int_{y_1}^{y_2} \mathcal{B}^2(y)dy \leq t\right\}\right| = o_p(1) \tag{26}$$

By Theorem 1:

$$\sup_{t\geq 0}\left|P\mathbb{1}\{T_n \leq t\} - P\mathbb{1}\left\{\int_{y_1}^{y_2} \mathcal{B}^2(y)dy \leq t\right\}\right| = o(1) \tag{27}$$

By our derivation above $T_n^* = \int_{y_1}^{y_2} \mathcal{B}_n^{*2}(y)dy + o_p(1)$ which implies:

$$\sup_{t\geq 0}\left|P\mathbb{1}\{T_n^* \leq t\} - P\mathbb{1}\left\{\int_{y_1}^{y_2} \mathcal{B}_n^{*2}(y)dy \leq t\right\}\right| = o(1) \tag{28}$$

Putting (26), (27) and (28) together and using Law of Iterated Expectations we obtain

$$\sup_{t\geq 0} |P\mathbb{1}\{T_n^* \leq t\} - P\mathbb{1}\{T_n \leq t\}| = o_p(1)$$

Now taking $t = c_\kappa^*$ concludes the proof.

A.4 Proof of Theorem 3

The proof follows lines similar to the proof of Theorem 2. We proceed in three steps. First we show bootstrap linear representation for the first step estimator $\beta^*(g)$ where $g \in \mathcal{N}_g$. Second, using this result we obtain similar representation for the bootstrap estimator of the Box-Cox parameter, $\hat{\gamma}$. In the third step we combine both results.

Step 1 Note that minimizing $S_n(g, b)$ is the same as minimizing $S_n(g, b) - S_n(g, \beta(g))$ and similarly for the bootstrap sample. Thus, we can write $S_n(g, b) = U_n^{(2)} h_\theta^1$ and $S_n^*(g, b) = U_n^{*(2)} h_\theta^1$ where

$$\begin{aligned} h_\theta^1(w_1, w_2) = {} & s(\Lambda(y^1, g) - \Lambda(c, g), \Lambda(y^2, g) - \Lambda(c, g), (x^1 - x^2)'b) \\ & - s(\Lambda(y^1, g) - \Lambda(c, g), \Lambda(y^2, g) - \Lambda(c, g), (x^1 - x^2)'\beta(g)). \end{aligned}$$

Additionally, define:

$$\tau_\theta^1(w) = E[h_\theta^1(w, W) + h_\theta^1(W, w)].$$

and let $\partial_\beta \tau_\theta^1$ be the gradient and let $\partial_\beta^2 \tau_\theta^1$ be the Hessian of τ_θ^1 with respect to β.

We can use Lemma 3 to show $P \sup_\theta |U_n^{*(2)} h_\theta^1 - P^2 h_\theta^1| \to 0$ and:

$$\begin{aligned} U_n^{(2)} h_\theta^1 &= (b - \beta(g))' 2 P_n \partial_\beta \tau_{(g,\beta(g))}^1 - \frac{1}{2}(b - \beta(g))' A_1(g)(b - \beta(g)) + o_p(\|(b - \beta(g))\|^2) + o_p(n^{-1}) \\ U_n^{*(2)} h_\theta^1 &= (b - \beta(g))' 2 P_n^* \partial_\beta \tau_{(g,\beta(g))}^1 - \frac{1}{2}(b - \beta(g))' A_1(g)(b - \beta(g)) + o_p(\|(b - \beta(g))\|^2) + o_p(n^{-1}) \end{aligned}$$

as $\theta = (g, b(g)) \to (g, \beta(g))$, uniformly over $g \in \mathcal{N}_g$, where $A = -P\partial_\beta^2 \tau_{(g,\beta(g))}^1$.

Let us verify the conditions of Lemma 3. Assumption (a) is satisfied with Θ_β and $\mathcal{N}_\gamma$. Continuity of $P^2 h_\theta^1$ for every $g \in \mathcal{N}_g$ follows from continuity of the function $s(\cdot, \cdot, \cdot)$. Part (b) follows from

Assumption 4(e), compactness of $\mathcal{N}_\gamma \times \Theta_\beta$, continuous differentiability of s, condition (c) in the statement of the theorem and Lemma 2.13 in Pakes & Pollard (1989). Parts (c) follows from Assumption 4(e), compactness of $\mathcal{N}_\gamma \times \Theta_\beta$ and condition (c) in the statement of the theorem. Note that Lemma A.1 in Chen (2012) (our assumptions imply his Assumptions 1-4) implies that $P\partial_\beta \tau^1_{(g,\beta(g))} = 0$ for all $g \in \mathcal{N}_\gamma$. Finally, part (d) follows from continuity of s, compactness of Θ_β and dominated convergence theorem.

The previous derivation implies that

$$S^*_n(g,b) = P^2 h^1_\theta(W_1, W_2) + o_p(1)$$

holds uniformly over $\theta \in \mathcal{N}_\gamma \times \Theta_\beta$. Chen (2012) shows that the expression on the right is uniquely minimized at $\beta(g)$ for any $g \in \mathcal{N}_\gamma$. It follows that $\beta^*(g)$ is consistent for $\beta(g)$ when $g \in \mathcal{N}_\gamma$. Proceeding as in the proof of Theorem 2 we obtain:

$$\beta^*(g) - \beta(g) = A_1^{-1}(g) 2 P^*_n \partial_\beta \tau^1_{(g,\beta(g))} + o_p(n^{-1/2}) \tag{29}$$

Additionally, Chen (2012) shows that the class of functions $\partial_\beta \tau^1_{(g,\beta(g))}$ is Euclidean with square integrable envelope, which implies:

$$\beta^*(g) - \beta(g) - [\beta^*(\gamma) - \beta(\gamma)] = o_p(n^{-1/2}) \tag{30}$$

for $g \in \mathcal{N}_\gamma$.

Step 2 Again, recenter R_n to $R_n(g,b) - R_n(\gamma,\beta)$ and similarly for the bootstrap sample. We have $R_n(g,b) = \int_c^\infty \int_c^\infty U_n^{(2)} h^{BC}_{\theta,y} d\Phi_1(y_1) d\Phi_2(y_2)$ and $R^*_n(g,b) = \int_c^\infty \int_c^\infty U_n^{*(2)} h^{BC}_{\theta,y} d\Phi_1(y_1) d\Phi_2(y_2)$. We can use Lemma 3 to show $P \sup_{\theta,y} |U_n^{*(2)} h^{BC}_{\theta,y} - P^2 h^{BC}_{\theta,y}| \to 0$ and:

$$\begin{aligned} U_n^{(2)} h^{BC}_{\theta,y} &= (\theta - \theta_0)' 2 P_n \partial \tau^{BC}_{\theta_0,y} - \frac{1}{2}(\theta-\theta_0)' A_{BC}(y)(\theta-\theta_0) + o_p(\|(\theta-\theta_0)\|^2) + o_p(n^{-1}) \\ U_n^{*(2)} h^{BC}_{\theta,y} &= (\theta - \theta_0)' 2 P^*_n \partial \tau^{BC}_{\theta_0,y} - \frac{1}{2}(\theta-\theta_0)' A_{BC}(y)(\theta-\theta_0) + o_p(\|(\theta-\theta_0)\|^2) + o_p(n^{-1}) \end{aligned} \tag{31}$$

as $\theta \to \theta_0$, uniformly over y, where $A_{BC}(y) = -P\partial^2 \tau^{BC}_{\theta,y}$.

Let us verify the conditions of Lemma 3. Assumption (a) is implied by condition (a) in the statement of the theorem. Continuity of $P^2 h^{BC}_{\theta,y}$ for every $y \in \mathcal{Y}$ follows from Assumptions 4(b),(e). Part (b) has been shown by Chen (2012). Parts (c) follows from Assumption 4(e), compactness of Θ and condition (c) in the statement of the theorem. Finally, part (d) follows from Assumption 4(b), continuity of Λ and argument similar to the proof of Theorem 4 in Sherman (1993), p.131.

Now

$$R^*_n(\theta) = \int_c^\infty \int_c^\infty P^2 h^{BC}_{\theta,y} d\Phi_1(y_1) d\Phi_2(y_2) + o_p(1)$$

holds uniformly over $\theta \in \Theta$. Chen (2012) shows that the expression on the right is uniquely minimized at $\theta_0 = (\gamma, \beta)$. Now following arguments in the proof of Theorem 1 in his paper we can show that $(\beta^*(\gamma^*), \gamma^*)$ is consistent for (γ, θ). Further following his argument and using (31) and (30) we arrive at:

$$R^*_n(g, \beta^*(g)) - R^*_n(\gamma, \beta^*(\gamma)) = (g-\gamma) P^*_n \Omega_\gamma - \frac{1}{2}(g-\gamma)^2 v_{BC} + o_p((g-\gamma)^2) + o_p(n^{-1})$$

where:

$$\Omega_\gamma = \begin{bmatrix} 1 \\ \frac{\mathrm{d}\beta(\gamma)}{\mathrm{d}g} \end{bmatrix}' 2 \left[\int_c^\infty \int_c^\infty \partial \tau^{BC}_{\theta_0,y} d\Phi_1(y_1) d\Phi_2(y_2) + V_{BC} \begin{bmatrix} 0 \\ A_1^{-1}(\gamma) \partial_\beta \tau^1_{\theta_0} \end{bmatrix} \right], \quad v_{BC} = \begin{bmatrix} 1 \\ \frac{\mathrm{d}\beta(\gamma)}{\mathrm{d}g} \end{bmatrix}' V_{BC} \begin{bmatrix} 1 \\ \frac{\mathrm{d}\beta(\gamma)}{\mathrm{d}g} \end{bmatrix}$$

which by the same argument as in the proof of Theorem 2 gives:

$$\gamma^* - \gamma = v^{-1}_{BC} P^*_n \Omega_\gamma + o_p(n^{-1/2}) \tag{32}$$

Step 3 Putting (30), (29) and (32) together we obtain:

$$\beta^*(\gamma^*) - \beta = P^*_n \left[A_1^{-1}(\gamma) \partial_\beta \tau^1_{\theta_0} + \frac{\mathrm{d}\beta_\gamma}{\mathrm{d}g} v^{-1}_{BC} \Omega_\gamma \right] + o_p(n^{-1/2}) \tag{33}$$

Now (33) and (32) imply:

$$P \left| \beta^* - \beta - P^*_n \left[A_1^{-1}(\gamma) \partial_\beta \tau^1_{\theta_0} + \frac{\mathrm{d}\beta_\gamma}{\mathrm{d}g} v^{-1}_{BC} \Omega_\gamma \right] \right| = o(n^{-1/2}) \quad \text{and} \quad P \left| \gamma^* - \gamma - v^{-1}_{BC} P^*_n \Omega_\gamma \right| = o(n^{-1/2}) \tag{34}$$

(see e.g. Theorem 1.4.C in Serfling (1980) where the uniform integrability follows from assumptions of the theorem). Finally, zero expectation and finite variance of the first term on the right hand side in (33) and (32) follow from Chen (2012) and our assumptions.

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